Conférence Universitaire de Suisse Occidentale (CUSO)

Singularities of Algebraic Surfaces

Lecture Notes

Lectures given by

Dr. Claudia Stadlmayr (Université de Neuchâtel)

as part of the CUSO Graduate Lectures
University of Bern, ExWi Building, Room B78
Fall Semester 2025 – starting September 24, 2025
Revised complete edition, November 2026

These lecture notes are prepared by Dr. Claudia Stadlmayr, Université de Neuchâtel, for the CUSO Graduate Lecture Series *Singularities of Algebraic Surfaces*, held weekly at the University of Bern. The lectures are given once per week (without separate exercise sessions) and form part of the CUSO Doctoral Program in Mathematics.

These notes accompany the twelve lectures. They are intended for participants of the course and for readers who wish to work through its examples and proofs independently.

Comments, suggestions for improvement, and reports of errors are very welcome.
Please send them to `claudia.stadlmayr@unine.ch` or `c.stadlmayr@gmx.net`.

Course information: https://www.cuso.ch/onepage/?p=5768&uid=7770

[...] si soleva dire che, mentre le curve algebriche (già composte in una teoria armonica) sono create da Dio, le superficie invece sono opera del Demonio. Ora si palesa invece che piacque a Dio di creare per le superficie un ordine di armonie più riposte ove rifulge una meravigliosa bellezza [...]

[...] it used to be said that while algebraic curves (already composed in a harmonic theory) are created by God, surfaces, instead, are the work of the devil. Now, on the contrary, it is clear that God chose to create for surfaces an order of more hidden harmonies, where a wonderful beauty shines forth [...]

(Federigo Enriques in [Enr49])

Contents

0 Overview/Introduction 6

1 Platonic solids and finite subgroups of SL_2 7

1.1 The five solids and their rotations 7
Counting a solid in two different ways 8

1.2 From rotations to matrices of determinant one 9
What quaternionic multiplication does to a vector 9

1.3 A count on the projective line 10
Recovering the orbits from the signature 12

1.4 Polynomials on an orbit space 13

2 Affine algebraic geometry and the spectrum of a ring . 17

2.1 Algebras and ideals 17
How a quotient ring recognizes a point 18

2.2 Noetherian rings and Hilbert's Nullstellensatz 19

2.3 Zariski topology 20
The line, the plane, and a family over the integers 21

2.4 Functoriality of the spectrum 22

2.5 Spectra and quotients of invariant rings 24
Following a closed point through two ring maps 25

3 Looking at one point 26

3.1 Functions, values and neighbourhoods 26
Why the extra factor occurs in equality of fractions 27

3.2 Dimension, height and transcendence degree 29
Two dimension counts on the same surface 30

3.3 Tangent spaces and regularity 31
Linear terms, derivations and minimal generators 32
The tangent cone remembers another order 33

3.4 Three ways for two arcs to meet 34

4 Completion and changes of formal coordinates 34

4.1 Keeping every infinitesimal neighbourhood 35

4.2 What completion preserves 36
Which information survives the inverse limit? 37

4.3 Inverting a substitution 38

4.4 Hensel lifting . . . 39
The Newton step without an analytic Taylor formula. . . . 40
4.5 Weierstrass preparation . . . 41
Projection and the meaning of the distinguished polynomial. . . . 42

5 Hypersurfaces and plane curve double points . . . 43
5.1 How many variables does a singularity need? . . . 43
Minimal equations and embedded components. . . . 44
5.2 Classification in multiplicity two . . . 44
5.3 Distinguishing the normal forms . . . 46
5.4 Four lines and a modulus . . . 47

6 Graded rings, projective space and the blow-up . . . 52
6.1 How to make directions into points . . . 52
6.2 The incidence variety and the Rees algebra . . . 54
6.3 Strict transform, total transform and the nodal cubic . . . 56
6.4 The first blow-up of a double point . . . 59
6.5 Recognizing a singular chart . . . 60

7 Normalization and resolution . . . 62
7.1 Putting missing integral functions back . . . 62
7.2 Normal surfaces can still be singular . . . 65
7.3 The surface resolution theorem . . . 68
A historical interlude. . . . 69
7.4 Blowing up the vertex of a cone . . . 69

8 Valuations, divisors and the canonical class . . . 73
8.1 The order along a curve . . . 73
8.2 Weil divisors, principal divisors and the class group . . . 73
Why the canonical divisor? . . . 76
8.3 Measuring the zeros and poles of a differential . . . 77
8.4 Three canonical divisors . . . 78

9 What the exceptional curves remember . . . 83
9.1 Pulling back the canonical divisor . . . 83
9.2 Intersections, minimality and the fundamental cycle . . . 84
9.3 Rationality and the ADE characterization . . . 86

What does the higher direct image measure? 87
9.4 The six curves of E_6 90
9.5 What changes in small characteristic 90
9.6 Back on a projective surface 92
9.7 A final look at type A_1 97

My poetry–math fun: Rilke poetry associated to each lecture 98

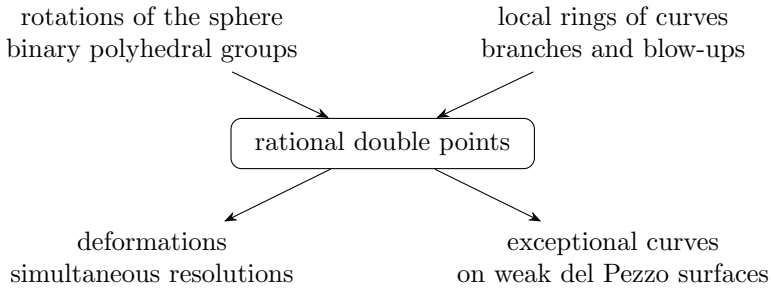
References 105

0. Overview/Introduction

How could looking at the rotations of an icosahedron tell us anything about a singular point on an algebraic surface? If one starts with the rotations, one obtains a ring of invariant polynomials and writes down an equation. If one starts with the singular point, one blows it up and draws a picture of the curves that lie above it. The fact that these two paths lead to the same list is the subject of these lectures.

We shall work out the cyclic example almost from the beginning. The equation $uv = w^2$ will then accompany us through the definitions of a local ring, a tangent space, a blow-up and a canonical divisor. In between, we shall spend some time with curves. There one can see with two coordinates and a sheet of paper what a formal change of variables does, and why a resolution remembers more than the picture of a tangent line.

The base field is indicated in the statements. In particular, the last part distinguishes what can be read from a resolution graph in all characteristics from what can be read from that graph about the equation. All rings are commutative with identity. For a first reading it is enough to know the algebra of ideals, fields and polynomial rings. Results from cohomology and intersection theory enter towards the end.



Starting at the upper left, lift the rotations to $\mathrm{SL}_2(\mathbb{C})$ and take invariant polynomials on $\mathbb{C}^2$. The relations among those invariants are equations of rational double points. The other upper arrow follows our route through curve germs: a node already lets us see what a blow-up does to tangent directions before we try it on a surface. Follow the lower left arrow to vary a singularity in a family and ask whether its resolutions fit together. Follow the lower right arrow to a weak del Pezzo surface: contracting its configurations of (-2)-curves produces singular points, and the intersections among the contracted curves give their ADE diagrams.

Marginal notes mark the ends of the twelve meetings. Each links to the corresponding poem in the last section. The table there gives the dates and links back to the mathematics.

1. Platonic solids and finite subgroups of SL_2

1.1. **The five solids and their rotations.** In Plato's *Timaeus*, the tetrahedron, octahedron, icosahedron and cube are associated with fire, air, water and earth. The fifth construction is reserved for the cosmos [Pla29, 53c–56c]. Book XIII of Euclid's *Elements* constructs the five regular solids in a sphere and ends by showing why there are only five [Euc56, Book XIII, Prop. 18 and scholium]. A later scholium attributes the tetrahedron, cube and dodecahedron to the Pythagoreans and the remaining two to Theaetetus. The evidence for the history of their discovery requires more care than the elementary proof of the list [Wat72]. In 1596 Kepler fitted the five solids between the six planetary spheres in his *Mysterium cosmographicum* [Kep96, Slo83]. What will concern us is another survival of the same five shapes, in the equations for rings of invariants in Klein's 1884 lectures on the icosahedron [Kle93].

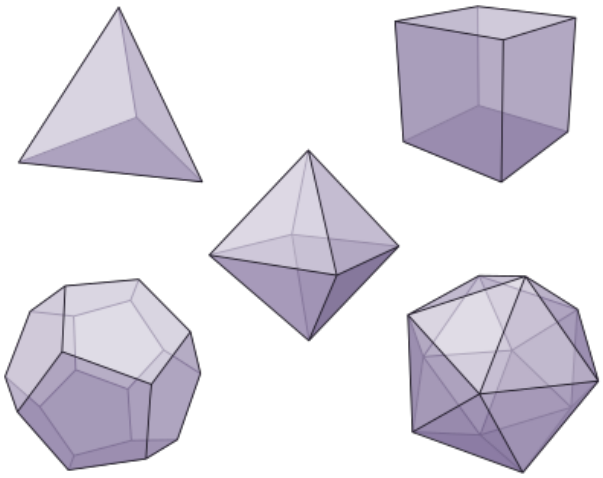


Figure 1. The five regular solids, as in the introductory notes. A cube and an octahedron have the same rotation axes, as do a dodecahedron and an icosahedron.

A convex regular polyhedron has congruent regular polygons for faces and the same vertex figure at every vertex. If q regular p-gons meet at a vertex, the sum of their angles is strictly less than 2π. Thus $p, q \geq 3$ and

$$q(1-2/p) < 2, \qquad 1/p + 1/q > 1/2.$$

The possibilities are $(3,3), (4,3), (3,4), (5,3), (3,5)$. In this order they give the tetrahedron, cube, octahedron, dodecahedron and icosahedron, shown in the preceding figure. Joining the centres of adjacent faces interchanges p and q. Taking the dual twice gives a homothetic copy of the original

regular solid, so the two solids, centred at the origin, have precisely the same rotation group.

Counting a solid in two different ways. An angle count finds the five candidates, but it says nothing yet about the number of their vertices. Denote the numbers of vertices, edges and faces by v, e, f. Each edge belongs to two faces and has two ends. Count the incidences between a face and one of its sides, and then between a vertex and one of its edges. This gives

$$pf = 2e, \qquad qv = 2e.$$

The surface of a convex polyhedron is a sphere, so $v - e + f = 2$. Eliminating v and f yields

$$e = \frac{2}{2/p + 2/q - 1}, \qquad v = \frac{2e}{q}, \qquad f = \frac{2e}{p}.$$

For a dodecahedron, for instance, $p = 5$ and $q = 3$. The denominator in the first formula is $1/15$, so there are thirty edges, twenty vertices and twelve faces. The numbers for the dual icosahedron are obtained simply by interchanging v and f.

	p	q	v	e	f
Tetrahedron	3	3	4	6	4
Cube	4	3	8	12	6
Octahedron	3	4	6	12	8
Dodecahedron	5	3	20	30	12
Icosahedron	3	5	12	30	20

Here is the group-theoretic content of the construction of the dual. If a rotation carries a face F of P to F', it carries the centre of F to the centre of F'. It also preserves the property of two faces having a common side. It therefore takes the vertices and edges of $P^\vee$ to vertices and edges of $P^\vee$. This gives an inclusion of rotation groups *inside the same* SO(3). Interchanging P and $P^\vee$ gives the reverse inclusion, since the common rescaling needed to identify $P^{\vee\vee}$ with P commutes with every rotation. This explains why an enumeration that begins with five solids leads to three polyhedral groups.

There is a quick check on the orders. Choose a face and mark an edge on it with the orientation induced by the outward normal of the face. Its stabilizer in the rotation group is trivial. Indeed fixing the oriented edge and the face normal fixes a basis of real three-space. Every such marking can be carried to every other one, first by moving the face and then by rotating it. Thus there are $pf = 2e$ rotations. A cube gives 24, and its dual must give the same answer. We next count by axes, since this second count tells us the *orders* of the elements.

Let T, O, I be the groups of rotations of the tetrahedron, octahedron and icosahedron. A nonidentity rotation fixes an axis through opposite

distinguished points of the solid. (For the tetrahedron a vertex is opposite a face.) Counting angles on these axes gives the following table [Lam86].

Solid	Axis through	Orders	Number of rotations
Tetrahedron	vertex	3	8
	edge midpoint	2	3
Octahedron	vertex	2, 4	3, 6 respectively
	face centre	3	8
	edge midpoint	2	6
Icosahedron	vertex	5	24
	face centre	3	20
	edge midpoint	2	15

Adding the identity gives $12, 24, 60$. The first group permutes the four vertices of the tetrahedron evenly. The second permutes the four body diagonals of the dual cube faithfully. Their orders therefore identify them with A_4 and S_4. The icosahedral group acts on the five inscribed cubes of the dual dodecahedron and identifies with A_5 [Lam86].

1.2. **From rotations to matrices of determinant one.** To turn a rotation in three real dimensions into a linear substitution in two complex variables, we use the quaternions. Every quaternion is $a + bi + cj + dk$, where $i^2 = j^2 = k^2 = ijk = -1$. The following representation is a homomorphism of real algebras. The four entries recover a, b, c, d, so it is injective.

$$\Phi : \mathbb{H} \longrightarrow M_2(\mathbb{C}), \qquad a + bi + cj + dk \longmapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

In fact $\Phi(i)\Phi(j) = \Phi(k) = -\Phi(j)\Phi(i)$, and the determinant is $a^2 + b^2 + c^2 + d^2$. The unit quaternions, sometimes called versors, map onto SU(2) since every special unitary matrix has the form $\left(\begin{smallmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{smallmatrix}\right)$ with $|\alpha|^2 + |\beta|^2 = 1$.

What quaternionic multiplication does to a vector. There are two different kernels one could ask about here. For the map of algebras Φ, an element of the kernel has all four matrix entries zero. The real and imaginary parts of its first row give $a = b = c = d = 0$. On the group of units the neutral element is the identity matrix. A quaternion whose matrix is the identity is therefore 1. The double covering will arise only at the next step, when a unit acts by conjugation.

Write $\bar q = a - bi - cj - dk$. Then $q\bar q = a^2 + b^2 + c^2 + d^2$ and $\overline{qr} = \bar r \bar q$. If q is a unit quaternion, $q^{-1} = \bar q$. If v is imaginary, $\bar v = -v$, so

$$\overline{qvq^{-1}} = q\bar v q^{-1} = -qvq^{-1}.$$

We have proved directly that conjugation carries the imaginary subspace to itself. Multiplicativity of the norm proves that it preserves lengths. To see the angle, pick a unit imaginary u and a unit imaginary v perpendicular to u. In the basis i, j, k, multiplication of imaginary vectors has real part

minus their scalar product and imaginary part their cross product. Hence $uv = -vu$ and $uvu = v$. Putting $q = c + su$ with $c^2 + s^2 = 1$, one obtains

$$\begin{aligned} qvq^{-1} &= (c + su)v(c - su) \\ &= c^2 v + cs(uv - vu) - s^2 uvu \\ &= (c^2 - s^2)v + 2cs\, uv. \end{aligned}$$

Take $c = \cos\alpha$, $s = \sin\alpha$. The last expression is a rotation through 2α in the oriented plane with basis v, uv. The vector u itself is fixed. In particular $q = i$ fixes i and sends j, k to $-j, -k$. This is a half-turn below and an element of order four above. Both q and $-q$ give this same half-turn.

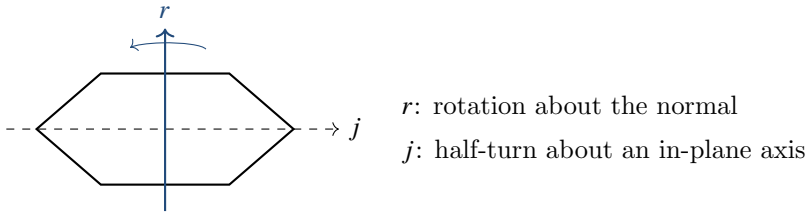


FIGURE 2. Two spatial rotations preserving a regular polygon. A half-turn about the horizontal axis reverses the cyclic order of its vertices.

Lemma 1.1 (Quaternionic double covering)**.** *Conjugation by unit quaternions induces a surjective homomorphism* $\mathrm{SU}(2) \to \mathrm{SO}(3)$ *with kernel* $\{I, -I\}$.

Proof. Now let a unit quaternion act on $\mathrm{Im}\,\mathbb{H} = \langle i, j, k\rangle_{\mathbb{R}}$ by $v \mapsto qvq^{-1}$. It preserves this subspace and its Euclidean norm. The determinant is positive by connectedness of S^3. More explicitly, if u is a unit imaginary quaternion and $q = \cos(\theta/2) + u\sin(\theta/2)$, conjugation fixes u and rotates $u^\perp$ by θ. This gives every rotation. The kernel consists of the central unit quaternions $\{\pm 1\}$. We obtain

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1.$$

Taking inverse images doubles the group orders. Conversely a finite group in $\mathrm{SL}_2(\mathbb{C})$ preserves a positive Hermitian form, obtained by summing the translates of an arbitrary one. In an orthonormal basis it lies in $\mathrm{SU}(2)$. □

1.3. **A count on the projective line.**

Theorem 1.2. *Let k be an algebraically closed field and let $G \subseteq \mathrm{SL}_2(k)$ be a finite subgroup with $\mathrm{char}(k) \nmid |G|$. Then, G is conjugate to one of the following:*

(1) The cyclic group C_N *of order N generated by a matrix of the form*

$$\begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix},$$

where ζ_N is a primitive N-th root of unity.

(2) *The* binary dihedral group BD_{4N} *of order* $4N$ *with* $N \geq 2$ *generated by the group* C_{2N} *above and the matrix*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(3) *The* binary tetrahedral group BT_{24} *of order* 24 *generated by the group* BD_8 *above and the matrix*

$$\frac{1}{2}\begin{pmatrix} 1+\sqrt{-1} & -1+\sqrt{-1} \\ 1+\sqrt{-1} & 1-\sqrt{-1} \end{pmatrix}.$$

(4) *The* binary octahedral group BO_{48} *of order* 48 *generated by the groups* BD_{16} *and* BT_{24} *above.*

(5) *The* binary icosahedral group BI_{120} *of order* 120 *generated by* BD_{20} *as above and the matrix*

$$\frac{1}{\sqrt{5}}\begin{pmatrix} -\zeta_5+\zeta_5^4 & \zeta_5^2-\zeta_5^3 \\ \zeta_5^2-\zeta_5^3 & \zeta_5-\zeta_5^4 \end{pmatrix}.$$

Proof. The tame projective classification, including the conjugacy assertion, is proved in [Bea10, §§1–3]. Over $\mathbb{C}$ the determinant-one lifts and their explicit generators are given in [Kle93, Part I, Chs. I–II]. We give the orbit count and lifting argument below. The diagonal cyclic groups in the statement include the groups of odd order, for which passage to the projective group has trivial kernel. □

Here C_{2N} denotes the diagonal subgroup defined with a primitive $2N$-th root. In the dihedral case the second generator b satisfies $b^2 = -I$ and $bab^{-1} = a^{-1}$ for a diagonal generator a. In the exceptional cases the choices of primitive roots and square roots are made compatibly with the displayed representatives. Over $\mathbb{C}$ the three groups are precisely the inverse images of T, O, I.

For a regular N-gon in a horizontal plane in $\mathbb{R}^3$, the rotations that preserve its orientation in that plane form a cyclic group, rotating about the vertical axis. A half-turn about an axis *in* the plane induces a reflection of the polygon. Adding it gives its full dihedral symmetry group, now realized entirely by spatial rotations. Upstairs in $\mathrm{SU}(2)$, the lift of the half-turn has order four: applying it twice gives $-I$, which acts trivially on imaginary quaternions. This accounts for the sign in the matrix identity $b^2 = -I$.

Here is the counting argument behind the list. The map $\mathrm{SL}_2(k) \to \mathrm{PGL}_2(k)$ is onto, since a scalar multiple of any matrix has determinant one over an algebraically closed field. Its kernel on k-points is $\{\pm I\}$ (a singleton in characteristic two). Over $\mathbb{R}$ the sign of the determinant would prevent this argument. Let H be a nontrivial tame projective subgroup of order h. On the affine line a matrix acts by $z \mapsto (az+b)/(cz+d)$; its fixed points on the projective line are precisely its eigenlines. A nonidentity

projective element of order s has a representative whose sth power is the identity. The polynomial $T^s - 1$ is separable, so it has two distinct eigenlines. A point stabilizer is cyclic: after moving the point to infinity its elements are $z \mapsto az + b$, and the multiplier map to $k^\times$ is injective on the stabilizer, since its kernel consists of translations and the action is tame.

Let $e_1, \dots, e_r$ be the stabilizer orders of representatives of the orbits with nontrivial stabilizers. Count pairs (P, g) with $g \neq 1$ and $gP = P$. Counting by g gives $2(h-1)$. The orbit with stabilizer e_i contributes $(h/e_i)(e_i - 1)$ pairs. Consequently

$$\sum_i (1 - 1/e_i) = 2 - 2/h.$$

Since $2 \leq e_i \leq h$, one obtains $2 \leq r \leq 3$. For $r = 2$, both $e_i = h$, and after moving the two fixed points to $0, \infty$ the action is diagonal. For $r = 3$, order the e_i increasingly. Their reciprocals sum to more than one, so $e_1 = 2$. If $e_2 = 2$ we obtain $(2, 2, s)$ with $h = 2s$. Otherwise $e_2 = 3$ and $e_3 = 3, 4, 5$, giving $h = 12, 24, 60$. In the first case the orbit of size two can be moved to $\{0, \infty\}$; in suitable coordinates the action is generated by $z \mapsto \zeta_s z$ and $z \mapsto 1/z$.

For $(2, 3, 3)$ an orbit of length four gives a faithful permutation action on four letters, hence $H \simeq A_4$. Faithfulness follows because a projectivity fixing three points is the identity. To see the coordinates in this case, the commuting involutions in its normal Klein four group may be written $z \mapsto -z$, $z \mapsto 1/z$ and $z \mapsto -1/z$. Their fixed points are $0, \infty, \pm 1, \pm i$. An element of order three cyclically permutes the three pairs. Lifting these transformations yields the binary tetrahedral group above. The signatures $(2, 3, 4)$ and $(2, 3, 5)$ yield S_4 and A_5; for existence and uniqueness up to projective conjugacy see [Bea10, Dol12]. In odd characteristic a noncyclic group in the list contains $-I$: every lift of a nontrivial projective involution to SL_2 squares to $-I$. It must therefore be the full inverse image. In characteristic two all three-orbit cases are excluded by tameness.

Recovering the orbits from the signature. The count gives more than the group order. If the stabilizer has order e, precisely h/e directions make up that orbit. For the three rotation groups this recovers the actual distinguished points on the unit sphere:

Group	Signature	Orbit lengths	Directions
A_4	$(2, 3, 3)$	$6, 4, 4$	edge midpoints, vertices, face centres
S_4	$(2, 3, 4)$	$12, 8, 6$	edges, faces, vertices of an octahedron
A_5	$(2, 3, 5)$	$30, 20, 12$	edges, faces, vertices of an icosahedron

For the tetrahedron a vertex and the centre of the opposite face have stabilizers of the same order. Why are they two orbits? The rotations permute the four vertices, and no rotation preserving the solid can turn a vertex into a face centre. Thus the repeated 3 in the signature has geometric meaning.

For the cyclic case move the two points fixed by the whole group to $0, \infty$. The transformations are $z \mapsto az$. For the dihedral case these two points form a single orbit. Its pointwise stabilizer consists of multiplications by powers of ζ_s, while an element interchanging them is $z \mapsto b/z$. The substitution $w = cz$ changes its coefficient to bc^2, so over an algebraically closed field one can arrange that $b = 1$. In these coordinates the relations are visible:

$$r(z) = \zeta_s z, \qquad j(z) = 1/z, \qquad j^2 = 1, \qquad jrj = r^{-1}.$$

The projective order of an element has to be read modulo scalars. For example $\mathrm{diag}(\eta, \eta^{-1})$ acts by $z \mapsto \eta^2 z$. To get a projective rotation of even order s its lift has order $2s$. If s is odd one of its two lifts has order s.

Over $\mathbb{C}$ this observation can also be expressed in terms of traces. For a primitive rotation angle $2\pi/s$ the eigenvalues of a suitable lift to $\mathrm{SU}(2)$ are $\exp(\pm\pi i/s)$. Their product is one, and their sum is $2\cos(\pi/s)$. Choosing the other lift changes the sign of the trace. The projective order is recovered from the ratio of the eigenvalues. The ratio formulation still makes sense in arbitrary tame characteristic. For contrast, in characteristic p the substitution $z \mapsto z + 1$ has order p but only one fixed point, ∞. This example pinpoints exactly where the fixed-point count used tameness.

Example 1.3 (An order-three transformation in the tetrahedral group)**.** Choose $i^2 = -1$. On the Riemann sphere put $a(z) = -z$, $b(z) = 1/z$. These generate a Klein four group whose three nontrivial elements have fixed pairs $\{0, \infty\}$, $\{1, -1\}$ and $\{i, -i\}$. The transformation

$$s(z) = \frac{z + i}{z - i}$$

takes $0, \infty$ to $-1, 1$, then $-1, 1$ to $-i, i$, and then $-i, i$ to $0, \infty$. Checking three individual images shows that $s^3 = 1$. It permutes the three involutions by conjugation, so adjoining it to the four group gives the semidirect product of order twelve, isomorphic to A_4. One can see the normal subgroup in the tetrahedron itself: its three half-turns about opposite edge midpoints are exactly the double transpositions of the four vertices.

1.4. **Polynomials on an orbit space.** Now, consider the polynomial ring $k[x, y]$ in two variables over an algebraically closed field k. Every $f \in k[x, y]$ defines a function

$$\begin{aligned} f : \quad k^2 \quad &\to k \\ (a, b) \quad &\mapsto f(a, b). \end{aligned}$$

Given a finite subgroup $G \subseteq \mathrm{SL}_2(k)$, there is a natural G-action on k^2 and we can define the quotient space

$$k^2/G \;=\; \{\, G{\cdot}v \mid v \in k^2 \,\}$$

(as a *set* of orbits, for now).

Note that a polynomial $f \in k[x,y]$ does not necessarily give a well-defined function on k^2/G, because it may have different values on vectors in the same orbit.

Example 1.4. Assume that $\operatorname{char} k \neq 2$, and let $G = \{\pm 1\}$ act on k^2 by scalar multiplication. Then $[(1,0)] = [(-1,0)]$ in k^2/G, but the polynomial $f = x \in k[x,y]$ satisfies

$$f(1,0) = 1 \neq -1 = f(-1,0).$$

Thus f does *not* descend to a well-defined function on k^2/G.

The example shows that not all polynomials yield well-defined functions on the orbit space. In this case, for instance, the polynomials $f = x^2, xy$, and y^2 do define functions on k^2/G. In general, the condition for a polynomial to descend to the orbit space is precisely that it is an *invariant polynomial*. We now define the invariant ring inside the polynomial ring in two variables.

Definition 1.5. The *invariant ring with respect to* G is

$$k[x,y]^G := \{\, f \in k[x,y] \mid f(g \cdot (a,b)) = f(a,b) \ \text{ for all } g \in G,\ (a,b) \in k^2 \,\},$$

and this ring serves as a ring of functions on k^2/G.

Theorem 1.6 (Klein)**.** *For the tame groups above the invariant algebra is generated by three elements with one relation. The following are equations in suitable generators; the generator degrees refer to their degrees in* $k[x,y]$ [Kle93, Slo80b, Lam86].

Group	*Degrees (unordered)*	*Relation*	*Type*
C_N	$2, N, N$	$Z^N + XY$	A_{N-1}
BD_{4N}	$4, 2N, 2N+2$	$Z^2 + X^2Y + Y^{N+1}$	D_{N+2}
BT_{24}	$6, 8, 12$	$Z^2 + X^3 + Y^4$	E_6
BO_{48}	$8, 12, 18$	$Z^2 + X^3 + XY^3$	E_7
BI_{120}	$12, 20, 30$	$Z^2 + X^3 + Y^5$	E_8

The case $N = 1$ *is smooth.*

Proof. We prove the cyclic case in full. The remaining cases follow the same idea, although determining the generators requires more computation.

Step 1. Generators. Let $C_N = \langle \gamma \rangle$ act on k^2 by

$$\gamma \cdot (x,y) = (\zeta_N x,\ \zeta_N^{-1} y),$$

where $\zeta_N \in k$ is a primitive N-th root of unity (possible since k is algebraically closed and $\operatorname{char}(k) \nmid N$). Clearly x^N, y^N, and xy are invariant, hence so is the k-subalgebra they generate:

$$k[x^N, y^N, xy] \ \subseteq\ k[x,y]^{C_N}.$$

For the converse inclusion, let $f(x,y) = \sum_{i,j} a_{ij} x^i y^j \in k[x,y]^{C_N}$ be invariant. Then

$$f(x,y) = f(\zeta_N x, \zeta_N^{-1} y) = \sum_{i,j} a_{ij}\, \zeta_N^{i-j}\, x^i y^j.$$

Comparing coefficients gives $a_{ij} = a_{ij}\, \zeta_N^{i-j}$ for all i, j, hence $a_{ij} = 0$ whenever $N \nmid (i-j)$. Thus every monomial appearing in f has $i - j \equiv 0 \pmod{N}$, so f is a polynomial in x^N, y^N, and xy. Therefore $k[x,y]^{C_N} \subseteq k[x^N, y^N, xy]$, proving the equality.

Step 2. The relation. Define a surjective k-algebra homomorphism

$$\pi : \; k[X,Y,Z] \;\longrightarrow\; k[x^N, y^N, xy], \qquad X \mapsto x^N, \quad Y \mapsto -y^N, \quad Z \mapsto xy,$$

so that $\pi(Z^N + XY) = (xy)^N + x^N(-y^N) = 0$. Hence $(Z^N + XY) \subseteq \ker \pi$. We prove the reverse inclusion elementarily by considering the Z-degree. Write $\deg_Z(\cdot)$ for the degree in the indeterminate Z.

Step 2.1. Linear independence. Suppose that $G \in \ker \pi$ has Z-degree less than N. Write

$$G = \sum_{a,b,c} \alpha_{a,b,c}\, X^a Y^b Z^c, \qquad 0 \le c < N, \quad \alpha_{a,b,c} \in k,$$

a finite sum. Then

$$\pi(X^a Y^b Z^c) = (x^N)^a \cdot (-y^N)^b \cdot (xy)^c = (-1)^b\, x^{aN+c}\, y^{bN+c}.$$

For $0 \le c, c' < N$, the exponent pairs $(aN + c,\ bN + c)$ are pairwise distinct: if

$$aN + c = a'N + c' \quad \text{and} \quad bN + c = b'N + c'$$

with $0 \le c, c' < N$, then $N(a - a') = c' - c = N(b - b')$, so $c' = c$ and hence $a = a'$, $b = b'$. Therefore the monomials $\{x^{aN+c} y^{bN+c}\}_{a,b,c}$ occurring in $\pi(G)$ are k-linearly independent, forcing all $\alpha_{a,b,c} = 0$ when $\pi(G) = 0$. Thus $G = 0$.

Step 2.2. Division by the relation. Set $F := Z^N + XY$. Since F is monic in Z, the univariate division algorithm in $k[X,Y][Z]$ yields for any $H \in k[X,Y,Z]$ unique $Q, R \in k[X,Y,Z]$ with

$$H = QF + R, \qquad \deg_Z R < N.$$

Take $H \in \ker \pi$. Applying π gives

$$0 = \pi(H) = \pi(Q)\, \pi(F) + \pi(R) = 0 + \pi(R) = \pi(R).$$

By the preceding argument, $R = 0$, hence $H = QF \in (F)$. Therefore $\ker \pi \subseteq (F)$, and since $(F) \subseteq \ker \pi$ already, we have $\ker \pi = (F)$.

By the First Isomorphism Theorem,

$$k[X,Y,Z]/(Z^N + XY) \;\cong\; \mathrm{im}(\pi) = k[x^N, y^N, xy],$$

as claimed. □

The dihedral computation also makes visible a small trap in guessing the generators. First take invariants by C_{2N}, putting $a = x^{2N}, b = y^{2N}, c = xy$. Then $ab = c^{2N}$. The extra generator interchanges a, b and negates c. Thus $a - b$ and c separately change sign. The invariants are generated by

$$U = a + b, \qquad V = c^2, \qquad W = c(a - b).$$

Indeed, writing polynomials in $a+b, a-b, c$, the even monomials in the last two variables are generated by their squares and product, and $(a - b)^2 = U^2 - 4V^N$. Hence $W^2 = V(U^2 - 4V^N)$. Rescaling gives the equation in the table.

For comparison, over $\mathbb{C}$ put $F = xy(x^4 - y^4)$, $H = x^8 + 14x^4y^4 + y^8$ and $T = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$. The tetrahedral invariants are generated by F, H, T, with $H^3 - T^2 = 108F^4$. Passing to the octahedral group changes the signs of both F and T and fixes H. Hence H, F^2, FT give the degrees $8, 12, 18$.

Exercise 1.7 (Obtaining the octahedral equation from the tetrahedron)**.** Use the index-two inclusion of the binary tetrahedral group in the binary octahedral group to determine the relation among $X = F^2$, $Y = H$ and $Z = FT$ over $\mathbb{C}$.

Solution. The remaining involution acts on F, T by changing both signs. Averaging a polynomial representative with its transform shows that every invariant can be represented using monomials of even total degree in F, T. Such monomials are products of F^2, FT, T^2. The tetrahedral identity eliminates $T^2 = H^3 - 108F^4$, and multiplication by F^2 gives

$$Z^2 = XY^3 - 108X^3.$$

There is no further relation. Indeed the Jacobian determinant of (F, H) at $(1, 0)$ is -8, so these two binary forms are algebraically independent, as are X, Y. In the rational function field $\mathbb{C}(X, Y)$ the right-hand side has order one at $X = 0$ and therefore is not a square. Hence the displayed quadratic is the minimal polynomial of Z over that field. Monic division shows that it generates the whole kernel over $\mathbb{C}[X, Y]$ as well. Rescaling the variables gives the E_7 equation of the table, and each term has degree 36 in the original binary variables. □

For the icosahedron the polynomial $f = xy(x^{10} + 11x^5y^5 - y^{10})$ cuts out the twelve vertex directions on $\mathbb{P}^1$. There is a way to obtain the other two invariants without guessing them. The determinant of the Hessian of an invariant binary form is invariant under substitutions of determinant one, by the chain rule. The same holds for the Jacobian determinant of two invariants. Put

$$h = (f_{xx}f_{yy} - f_{xy}^2)/121, \qquad t = (f_x h_y - f_y h_x)/20.$$

In the abbreviations $u = x^5$, $v = y^5$ the expansion is

$$h = -(u^4 + v^4) + 228(u^3v - uv^3) - 494u^2v^2,$$
$$t = u^6 + v^6 + 522(u^5v - uv^5) - 10005(u^4v^2 + u^2v^4).$$

Substitution gives $t^2 + h^3 = 1728 f^5$. Each term has degree sixty. This is the E_8 equation after rescaling; its exponents $2, 3, 5$ are already the orders we encountered at an edge midpoint, a face centre and a vertex. The Hessian computation here is over $\mathbb{C}$. Certain normalizing constants vanish in positive characteristic, so these particular scaled formulas should be read there only after checking the scalars. We shall recover the exponents from a different calculation on the resolution.

End of lecture 1

2. Affine algebraic geometry and the spectrum of a ring

We have written down equations for quotient surfaces. What space do these equations describe? Over an algebraically closed field, the closed points of the quadric cone will indeed be the orbits of the involution on the plane. We first need a topology that remembers the curves lying on the cone as well as its ordinary points.

2.1. Algebras and ideals.

For the goal described above, we first have to recall the basic algebraic notions from which we want to build geometry. Ideals will correspond to equations that *cut out* so-called *varieties* as *vanishing sets*. Prime ideals and maximal ideals will correspond to points. We will make this precise in what follows.

First, we recall the concept of algebras and ideals:

Definition 2.1. Let A be a ring.

- An A-algebra is a ring B together with a ring homomorphism $A \to B$.
- An *ideal* of A is a subgroup $I \subseteq A$ of the additive group underlying A and such that $ab \in I$ for all $a \in A$ and $b \in I$.
- Given $a_1, \dots, a_n \in A$, the *ideal generated by* $a_1, \dots, a_n$ is

  $$(a_1, \dots, a_n) \coloneqq (a_1, \dots, a_n)A \coloneqq \{\sum_{i=1}^{n} b_i a_i \mid b_i \in A\}.$$

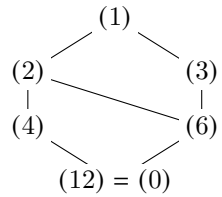
  It is the smallest ideal containing all the a_i.
- An ideal $I \subseteq A$ is called *prime ideal* if $I \neq A$ and $ab \in I$ implies $a \in I$ or $b \in I$. We usually denote prime ideals by $\mathfrak{p}$.
- The set of all prime ideals of A is called *spectrum* of A and denoted $\operatorname{Spec} A$.
- An ideal $I \subseteq A$ is called *maximal* if $I \neq A$ and I is maximal (with respect to inclusion) among all ideals with this property. We usually denote maximal ideals by $\mathfrak{m}$.
- The set of all maximal ideals of A is called *maximal spectrum* of A and denoted $\operatorname{Spec}_{\max} A$.

Remark 2.2. We can characterize the three types of ideals we defined above in terms of the quotient A/I.

- The extra condition on a subgroup $I \subseteq A$ that makes it an ideal guarantees that the multiplication on A descends to a multiplication on A/I making the projection map $A \to A/I, a \mapsto \overline{a}$ a ring homomorphism.
- The condition that a proper ideal $\mathfrak{p} \subseteq A$ is prime is equivalent to saying that $A - \mathfrak{p}$ is a *multiplicative subset* (i.e., $1 \in A - \mathfrak{p}$ and $a, b \in A - \mathfrak{p}$ implies $ab \in A - \mathfrak{p}$). It is also equivalent to saying that the quotient ring $A/\mathfrak{p}$ is an *integral domain*, i.e., a ring with no non-zero *zero divisors*. A zero divisor annihilates a nonzero element.
- One can show (Exercise!) that $\mathfrak{m} \subseteq A$ is maximal if and only if the ring $A/\mathfrak{m}$ is a field, i.e., a ring such that every non-zero element is a *unit*, that is, invertible with respect to multiplication. In particular, every maximal ideal is prime.

How a quotient ring recognizes a point. Let us carry out the exercise about maximal ideals. If I is maximal and $a \notin I$, the ideal $I + (a)$ strictly contains I. Thus $1 = i + ba$ for some $i \in I$ and $b \in A$, which gives $\bar{b}\bar{a} = 1$ in A/I. Every nonzero residue class has an inverse. Conversely, if A/I is a field and J is an ideal properly containing I, choose $a \in J \setminus I$. The inverse of $\bar{a}$ lifts to some $b \in A$. Then $ab - 1 \in I \subset J$, and since $ab \in J$ it follows that $1 \in J$. Thus $J = A$. Likewise the equation $\bar{a}\bar{b} = 0$ translates precisely into $ab \in I$, explaining the characterization of a prime ideal by a domain as quotient.

Example 2.3 (The ideal diagram for $\mathbb{Z}/12\mathbb{Z}$)**.** The ideals of $\mathbb{Z}/12\mathbb{Z}$ correspond, by inverse image, to the ideals of $\mathbb{Z}$ containing (12), hence to the positive divisors of 12. Inclusion reverses divisibility. In the diagram a line leads upwards to a larger ideal. The two maximal ideals (2) and (3) are prime because their quotients are fields. The zero ideal is not prime since $3 \cdot 4 = 0$ although neither factor vanishes. The quotient by (4) is $\mathbb{Z}/4\mathbb{Z}$; the nilpotent element 2 shows directly that (4) is not prime either.



In $k[t]$ the division algorithm makes every ideal principal: in a nonzero ideal choose a nonzero polynomial of least degree and divide every other element by it. The remainder lies in the ideal and is too small in degree to be nonzero. The nonzero primes are generated by irreducible polynomials. If k is algebraically closed these are precisely $(t - a)$, and evaluation at a identifies $k[t]/(t - a)$ with k. The zero ideal is also prime, but has quotient

$k[t]$, which is not a field. Over $\mathbb{R}$, (t^2+1) is another maximal ideal. The residue of t there has square -1; the residue field of this closed point is $\mathbb{C}$. A point of a spectrum comes equipped with a field in which the functions can take their values.

In two variables (x, y) is still maximal, but cannot be principal. A common divisor of x and y in $k[x, y]$ is a unit, whereas (x, y) is proper. This elementary example explains why asking for all ideals to be principal would already discard the affine plane. Asking that every ideal be finitely generated allows exactly the sort of equations we can actually write down.

2.2. **Noetherian rings and Hilbert's Nullstellensatz.** For general rings, ideals may not be generated by finitely many elements. Those rings for which this holds deserve a name:

Definition 2.4. A ring A is called *Noetherian* if all ideals $I \subseteq A$ are finitely generated.

For polynomial rings over algebraically closed fields, we have the following two fundamental results of Hilbert on the structure of their ideals:

Theorem 2.5. *Let k be an algebraically closed field and $A = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k.*

- *Hilbert's basis theorem: The ring A is Noetherian.*
- *Hilbert's Nullstellensatz: Every maximal ideal $\mathfrak{m} \subseteq A$ is of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$. More precisely, the map*

$$\begin{array}{rcl} k^n & \to & \operatorname{Spec}_{\max} A \subseteq \operatorname{Spec} A \\ (a_1, \ldots, a_n) & \mapsto & (x_1 - a_1, \ldots, x_n - a_n) \end{array}$$

is a bijection. [1]

Proof. The ascending chain condition is equivalent to finite generation of all ideals. The union of an ascending chain is an ideal, and finitely many generators of that union occur at some finite stage. Conversely, starting with an ideal that cannot be finitely generated, successively choose an element outside the ideal generated so far to obtain a strictly ascending chain.

To see what happens on passing from R to $R[t]$, let $I \subset R[t]$ be an ideal, where R is Noetherian. Let $L_n \subset R$ consist of the coefficients of t^n of elements of I of degree at most n, including zero. It is an ideal of R. Multiplication by t gives $L_n \subset L_{n+1}$, so this chain stabilizes, say at N. For $0 \leq n \leq N$ choose finitely many polynomials of degree at most n realizing generators of L_n. We claim that these finitely many polynomials generate I. If $f \in I$ has degree $d \leq N$, subtract an R-linear combination of the chosen polynomials for d to kill its leading coefficient. If $d > N$, do the same with t^{d-N} times the polynomials for N. The degree decreases, and induction finishes the

[1] This is just one of many versions of the Nullstellensatz.

proof. This is the basis theorem, applied successively starting from the field k. For the closed-point assertion, a maximal quotient of $k[x_1, \ldots, x_n]$ is a field of finite type over k. Zariski's lemma makes it finite algebraic, hence equal to the algebraically closed field k. The images a_i of the variables give the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$, and conversely evaluation at any such tuple has field quotient. See [Har77, Ch. I, Theorem 1.3A and its corollaries]. □

The field hypothesis in the other part of Hilbert's theorem enters at a different step. If $\mathfrak{m}$ is maximal, $K = k[x_1, \ldots, x_n]/\mathfrak{m}$ is both a field and a finitely generated k-algebra. Zariski's lemma says that such a field is finite algebraic over k [AM69, Ch. 5]. When $k = \bar{k}$, the residue classes of the x_i are therefore elements a_i of k. The inclusion $(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}$ is now an equality since the ideal on the left already has field quotient. Over a nonclosed field the same argument gives a finite residue extension. Distinguishing this step from Noetherianity will matter when we later use the Jacobian criterion.

The bijection $k^n \to \operatorname{Spec}_{\max} A$ allows one to translate statements about vanishing sets of polynomials in k^n to statements about maximal ideals in the ring A. Suppose $J \subset A$ and a polynomial g vanishes at every common zero of J. Then $g \in \sqrt{J}$. Indeed, if its image in A/J were not nilpotent, the localization $(A/J)_g$ would be nonzero. It has the finite presentation $(A/J)[T]/(Tg - 1)$, so a maximal ideal gives a k-point by the theorem. At this point J vanishes while g is invertible, a contradiction. The reverse inclusion is immediate. Thus the set of closed points determines the radical of J. How much of the distinction between J and $\sqrt{J}$ is visible to geometry? The ring of the closed subscheme will remember it, even where its topology cannot.

2.3. **Zariski topology.**

Definition 2.6. Let A be a ring.

- Given an ideal $I \subseteq A$, the *vanishing set* of I is
$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subseteq \mathfrak{p}\} \subseteq \operatorname{Spec} A.$$
- The *Zariski topology* on $\operatorname{Spec} A$ is the topology whose closed subsets are exactly the $V(I)$.

From now on, we consider $\operatorname{Spec} A$ and $\operatorname{Spec}_{\max} A$ as topological spaces with the Zariski topology (and the induced subspace topology).

Exercise 2.7. Let A be a ring.

(1) Show that the Zariski topology is indeed a topology on $\operatorname{Spec} A$.
(2) Show that if $\mathfrak{p} \in \operatorname{Spec} A$ is a prime ideal, then the topological closure of $\{\mathfrak{p}\}$ in $\operatorname{Spec} A$ is exactly $V(\mathfrak{p})$. In particular, $\operatorname{Spec}_{\max} A \subseteq \operatorname{Spec} A$ is exactly the subset of closed points.

(3) Assume k is algebraically closed and $A = k[x_1, \ldots, x_n]$. Then we can use the bijection between k^n and the maximal spectrum of A of Theorem 2.5 to equip k^n with a Zariski topology. Show that a subset $X \subseteq k^n$ is closed with respect to this topology if and only if there is an ideal $I \subseteq A$ such that

$$X = \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

We call such an X *affine algebraic set*. If I is prime, we say that X is an *affine algebraic variety*.

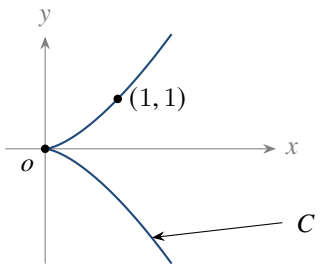
Solution. Use $V(1) = \varnothing$, $V(0) = \operatorname{Spec} A$, $\bigcap V(I_\alpha) = V(\sum I_\alpha)$ and $V(I) \cup V(J) = V(IJ)$. The last equality uses primeness. Among the closed sets containing $\mathfrak{p}$, the smallest is $V(\mathfrak{p})$. For example in $\operatorname{Spec} k[x, y]$ the point $(y - x^2)$ has as its closure the entire parabola. To specialize that one point to a particular point (a, a^2) is to enlarge its prime to $(x - a, y - a^2)$. □

The line, the plane, and a family over the integers. When $k = \bar{k}$, any proper closed subset of the affine line is finite. A nonzero polynomial has only finitely many roots, and conversely a finite set $\{a_1, \ldots, a_r\}$ is the vanishing set of $\prod_i (t - a_i)$. In the full spectrum one must add the point (0), whose closure is the whole line. Thus every nonempty open contains that one point. By contrast, in the affine plane the proper closed subset $V(y - x^2)$ contains infinitely many closed points even though it is cut out by a single irreducible equation.

Example 2.8 (The three kinds of points in a plane)**.** Let k be algebraically closed. In $\operatorname{Spec} k[x, y]$ consider the whole plane, the cuspidal curve $C = V(y^2 - x^3)$, and the point $o = (0, 0)$. Each has its own generic point, and specialization runs from left to right in

$$(0) \subsetneq (y^2 - x^3) \subsetneq (x, y).$$

The middle ideal is prime. Indeed the map $k[x, y] \to k[t]$ sending x to t^2 and y to t^3 has kernel $(y^2 - x^3)$: reduce a polynomial modulo the monic equation to $a(x) + yb(x)$ and compare the even and odd powers of t. The closure of the middle point consists of this generic point and all closed points $(x - a, y - b)$ with $b^2 = a^3$. The generic point itself has residue field $k(t)$, since $t = y/x$ in the fraction field. At o the residue field is k. This explains how the usual drawing can record a whole prime ideal in a single curve.



$\overline{\{(0)\}} = \mathbb{A}^2$
$\overline{\{(y^2 - x^3)\}} = C$
$\overline{\{(x, y)\}} = \{o\}$

FIGURE 3. Specialization in the affine plane. The curve is drawn on its real locus; the formulas on the right describe the three scheme-theoretic points.

The same construction for $A = \mathbb{Z}$ gives one closed point (p) for each positive prime and the generic point (0). The integer 6 takes value zero exactly at (2) and (3). At (5) its value is $1 \in \mathbb{F}_5$, and at the generic point it has value $6 \in \mathbb{Q}$. Thus a single element can be regarded as a function without choosing one common field for all of its values. The distinguished open $D(6)$ contains (0) and every closed point except $(2), (3)$. The functions on this open form $\mathbb{Z}[1/6]$. This is a model for the use of denominators in the next lecture.

For arbitrary A the operations on ideals and closed sets reverse inclusions. If W is a subset of $\operatorname{Spec} A$ put $I(W) = \bigcap_{\mathfrak{p} \in W} \mathfrak{p}$. Then

$$W \subseteq V(J) \quad \Longleftrightarrow \quad J \subseteq I(W).$$

In particular $\overline{W} = V(I(W))$, and $I(V(J)) = \sqrt{J}$. The latter identity for a spectrum holds over any ring: an element outside the radical survives in the nonzero localization $(A/J)_f$, which has a prime ideal. Its inverse image is a prime containing J and avoiding f. To recover the ideal using only k-valued points one needs the field and finiteness hypotheses of the Nullstellensatz.

2.4. **Functoriality of the spectrum.** The key reason for working with the spectrum instead of only the maximal spectrum is that preimages of prime ideals under ring homomorphisms are again prime. [2] This allows us to turn $\operatorname{Spec}(-)$ into a functor from the category of rings to the category of topological spaces:

Definition 2.9. (1) Let $\varphi : A \to B$ be a ring homomorphism. The *induced map on spectra* is the map

$$\begin{aligned} \varphi^* : \operatorname{Spec} B &\to \operatorname{Spec} A \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}). \end{aligned}$$

(2) If in addition φ is a k-linear morphism between finitely generated k-algebras, the above φ^* restricts to a morphism on maximal spectra:

$$\begin{aligned} \varphi^* : \operatorname{Spec}_{\max} B &\to \operatorname{Spec}_{\max} A \\ \mathfrak{m} &\mapsto \varphi^{-1}(\mathfrak{m}). \end{aligned}$$

Example 2.10. One of the key examples for an induced map on spectra is the one induced by the quotient map $\pi : A \to A/I$ by an ideal $I \subseteq A$. One can show that $\pi^* : \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ factors through a homeomorphism between $\operatorname{Spec}(A/I)$ and $V(I) \subseteq \operatorname{Spec} A$. In other words, prime ideals in A/I correspond bijectively in a natural way to prime ideals in A containing I.

[2] This is a nice small exercise.

One can check that the induced map on spectra is continuous with respect to the Zariski topology. In fact, for an ideal $I \subseteq A$, one has $(\varphi^*)^{-1}(V(I)) = V(\varphi(I)B)$.

Remark 2.11. Let us roughly explain what it means to be a functor and why taking the spectrum is an example:

- A *category* $\mathcal{C}$ is a collection of *objects* $\mathrm{Ob}(\mathcal{C})$ and for any two objects X, Y a collection of *arrows* $\mathrm{Mor}(X, Y)$ together with a composition law for arrows that is associative and has an identity.
- Given a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{op}$ is the category with the same objects but with reversed arrows.
- A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a law F that associates to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$ and to every arrow $f : X \to Y$ an arrow $F(f) : F(X) \to F(Y)$ such that identities and composition are preserved.
- The collections of rings and ring homomorphisms form a category Ring of rings and the collection of topological spaces and continuous maps form a category Top of topological spaces. The functor Spec $(-)$ associates to a ring A the topological space Spec A and to a ring homomorphism $\varphi : A \to B$ the induced map on spectra $\varphi^* : \mathrm{Spec}B \to \mathrm{Spec}A$. We observe that $\mathrm{id}^* = \mathrm{id}$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. This means exactly that Spec $(-)$ is a functor from Ring^{op} to Top.

Example 2.12 (Three ring maps, three pictures)**.** Take the maps $k[u] \to k[t]$, $u \mapsto t^2$, and $k[u] \to k$, $u \mapsto a$. The first map is injective. Its map of spectra is a double covering on the complement of $u = 0$ if k is algebraically closed of characteristic different from two. The second is surjective; its map of spectra includes one point. The tensor product describes the fibre of the first over the second:

$$k[t] \otimes_{k[u]} k \simeq k[t]/(t^2 - a).$$

For $a \neq 0$ the Chinese remainder theorem gives two copies of k; for $a = 0$ the result is $k[t]/(t^2)$. Its sole closed point accounts for its support, while the class of t records the multiplicity. Third, the localization $k[u] \to k[u, u^{-1}]$ identifies its spectrum with the complementary open of the origin. Although this map is injective too, the induced map misses a point. It is the nature of the ring extension, not injectivity by itself, that distinguished the first picture from the third.

These examples also explain the choice of prime ideals in the definition of Spec . For $\mathbb{Z} \hookrightarrow \mathbb{Q}$, the inverse image of the maximal ideal (0) is (0), which is not maximal in $\mathbb{Z}$. A map on maximal spectra for all rings would have nowhere to send this point. For a map of finitely generated algebras over a field an inverse image of a maximal ideal does remain maximal: the image of the source in the finite residue extension is an integral domain all

of whose elements are algebraic over the base field, and such a domain is a field.

2.5. **Spectra and quotients of invariant rings.**

Definition 2.13. Let A be a ring and G a finite group acting on A via ring homomorphisms. The associated *invariant ring* is the subring

$$A^G = \{a \in A \mid g(a) = a \text{ for all } g \in G\}.$$

We call $\operatorname{Spec} A^G$ the *quotient of* $\operatorname{Spec} A$ *by the action of* G.

Proposition 2.14. *Let a finite group G act by automorphisms on a ring A. The inclusion $A^G \hookrightarrow A$ induces a closed surjection $\operatorname{Spec} A \to \operatorname{Spec} A^G$, its fibres are the G-orbits, and it induces a homeomorphism $(\operatorname{Spec} A)/G \simeq \operatorname{Spec} A^G$ for the quotient topology.*

Proof. Every $a \in A$ satisfies the monic polynomial $\prod_{g \in G}(T - ga)$ over A^G. Lying over for integral extensions gives surjectivity, also after taking a quotient: the image of $V(I)$ is $V(I \cap A^G)$. Thus the map is closed. Points in one orbit have the same contraction. Conversely suppose that $\mathfrak{p}, \mathfrak{q}$ have the same contraction and that no translate of $\mathfrak{q}$ equals $\mathfrak{p}$. By incomparability for integral extensions, $\mathfrak{p}$ is contained in none of these translates. Finite prime avoidance gives $a \in \mathfrak{p}$ outside all of them. Its orbit product lies in $\mathfrak{p} \cap A^G$ and outside $\mathfrak{q}$, a contradiction. A closed surjection is a quotient map. See also [AM69, Ch. 5, Ex. 13] and [Gro71, Exp. V, Prop. 1.1]. □

There is a corresponding statement about morphisms. If $Z = \operatorname{Spec} B$ is affine, a morphism $\operatorname{Spec} A \to Z$ is G-invariant precisely when its ring map $B \to A$ takes values in A^G. The factorization $B \to A^G \hookrightarrow A$ is then unique. This is the universal property of the quotient for maps to affine schemes. In the case below, the three coordinate functions of the quotient are exactly the three generators of the invariant ring.

Example 2.15 (The quadric cone)**.** Let $k = \bar{k}$ have characteristic different from two. The quotient map for the sign action takes (s, t) to (s^2, t^2, st) in $V(UV - W^2)$. Given (u, v, w) on the cone with $u \neq 0$, choose s with $s^2 = u$ and set $t = w/s$; the two choices differ by simultaneous negation. If $u = 0$, then $w = 0$ and $t^2 = v$. Only the vertex has a single preimage on closed points. Scheme-theoretically the fibre there has ring $k[s, t]/(s^2, st, t^2)$; its length is three. Thus the statement about orbits is a statement about the underlying topological space. The structure sheaf remembers more.

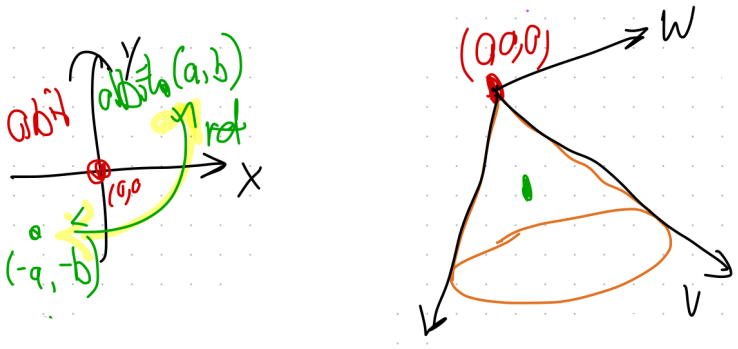


FIGURE 4. The sign quotient, drawn in the second lecture. The two green source points form one orbit. Its image is a single point on the quadric cone. Over the reals the drawing shows the sheet reached by real pairs.

Exercise 2.16 (Following a fibre down to the vertex)**.** On the chart $U \neq 0$ of the quadric cone, find a basis for the coordinate ring of the inverse image as a module over the base. Is the quotient map flat at the vertex?

Solution. On the target $V = W^2/U$, so the chart has ring $B = k[U, U^{-1}, W]$. On the source s is invertible, $s^2 = U$ and $t = W/s$. The inverse image has ring $B[S]/(S^2 - U)$ and basis $1, S$ by monic division. Moreover $2S$ is a unit, so this is a finite étale cover of degree two. Exchanging U and V covers the rest of the punctured cone. At the vertex the fibre ring instead has basis $1, s, t$. If the finite module were flat there, it would be free over the local ring, necessarily of rank two by passing to the fraction field. Reduction modulo the maximal ideal would give a two-dimensional fibre, a contradiction. The picture counts orbits; this calculation explains the extra length concentrated at the fixed point. □

Following a closed point through two ring maps. It is instructive to keep the coordinates in the plane separate from the coordinates in the ambient three-space. Take

$$k[U, V, W] \xrightarrow{\rho} B = k[U, V, W]/(UV - W^2) \xrightarrow{\iota} A = k[s, t].$$

The first map imposes the equation. The second sends the three residue classes to s^2, t^2, st and is injective by the cyclic calculation. For a point (a, b) of the plane compose with evaluation. A polynomial F maps to $F(a^2, b^2, ab)$, so its kernel is precisely

$$(U - a^2,\ V - b^2,\ W - ab).$$

Indeed this ideal is contained in the kernel and its quotient is k. The polynomial $UV - W^2$ belongs to it since $(a^2)(b^2) = (ab)^2$. Hence the point in three-space automatically lies on the surface. At the level of point sets

we have obtained the diagram

$$\mathbb{A}^2(k) \longrightarrow \mathbb{A}^2(k)/\{\pm 1\} \xrightarrow{\sim} V(UV - W^2)(k) \hookrightarrow \mathbb{A}^3(k).$$

The middle bijection uses the explicit reconstruction of s, t above. The stronger topological assertion follows from Proposition 2.14. In particular the first and last arrows are geometrically quite different, even though both come from evaluating polynomials.

End of lecture 2

3. Looking at one point

3.1. **Functions, values and neighbourhoods.** There is one qualification to the expression "ring of functions" which becomes important as soon as we look near a point. At $\mathfrak{p} \in \operatorname{Spec} A$, the value of $a \in A$ lies in the residue field $\kappa(\mathfrak{p}) = \operatorname{Frac}(A/\mathfrak{p})$ and is the image of a there. In particular $a(\mathfrak{p}) = 0$ exactly when $a \in \mathfrak{p}$. If $A = k[x, y]$ and $\mathfrak{p} = (y - x^2)$, the value of x at the generic point of the parabola is the indeterminate $x \in k(x)$. By contrast, at $(x - c, y - c^2)$ it is the scalar c. Functions vanishing on a subset W form the ideal $\bigcap_{\mathfrak{p} \in W} \mathfrak{p}$. Applied to $V(I)$ this gives $\sqrt{I}$. In particular nilpotents vanish at every point. The element ϵ in $k[\epsilon]/(\epsilon^2)$ is a good reason for keeping the ring as well as its point values.

The points belonging to every open neighbourhood of $\mathfrak{p}$ are precisely its generalizations $\mathfrak{q} \subseteq \mathfrak{p}$. Indeed the basic open $D(a)$ contains $\mathfrak{p}$ exactly when $a \notin \mathfrak{p}$. Requiring every such $D(a)$ to contain $\mathfrak{q}$ says exactly that $\mathfrak{q} \subseteq \mathfrak{p}$. We can realize this intersection of neighbourhoods as the spectrum of a single ring.

This will be a special case of a construction called localization with respect to a multiplicative subset.

Definition 3.1. Let A be a ring. A *multiplicative subset* of A is a subset $S \subseteq A$ such that $1 \in S$ and such that for all $a, b \in S$ also $ab \in S$.

Example 3.2. The two key examples of multiplicative subsets are the following:

(1) If $a \in A$, then $S = \{a^n\}_{n \geq 0} \subseteq A$ is multiplicative.
(2) If $\mathfrak{p} \in \operatorname{Spec} A$, then $S = A - \mathfrak{p} \subseteq A$ is multiplicative, by the defining property of prime ideals.

Definition 3.3. Let A be a ring and $S \subseteq A$ a multiplicative subset. The *localization of A with respect to S* is the set

$$S^{-1}A = \{\frac{a}{b} \mid a \in A, b \in S\}/\sim,$$

where

$$\frac{a}{b} \sim \frac{c}{d} \iff \exists e \in S : e(ad - bc) = 0 \in A.$$

together with the following addition and multiplication

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &\coloneqq \frac{ad+bc}{bd} \\ \frac{a}{b} \cdot \frac{c}{d} &\coloneqq \frac{ac}{bd}. \end{aligned}$$

If $S = \{a^n\}_{n\geq 0} \subseteq A$ for an element $a \in A$, we write $A_a \coloneqq S^{-1}A$ and if $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, we write $A_\mathfrak{p} \coloneqq S^{-1}A$.

Remark 3.4. One checks that $S^{-1}A$ with $+$ and $\cdot$ as defined above is indeed a ring and that the map $a \mapsto \frac{a}{1}$ defines a ring homomorphism. The induced map on spectra $\operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ factors through a homeomorphism with

$$X = \{\mathfrak{q} \in \operatorname{Spec} A \mid S \cap \mathfrak{q} = \emptyset\}.$$

equipped with the subset topology.

In particular, if $S = A - \mathfrak{p}$, then $X = \{\mathfrak{q} : \mathfrak{q} \subseteq \mathfrak{p}\}$, the intersection of neighbourhoods described above. This set contains a unique closed point, hence so does $\operatorname{Spec} A_\mathfrak{p}$, and one can check that the corresponding maximal ideal in $\operatorname{Spec} A_\mathfrak{p}$ is the ideal $\mathfrak{p}A_\mathfrak{p}$ generated by $\mathfrak{p}$.

Why the extra factor occurs in equality of fractions. In a domain and with zero excluded from S we may cross-multiply as usual. In a ring with zero divisors, elements of S can kill other elements before they become units. In $A = k[x,y]/(xy)$ invert x. We have $xy = 0$, so $y/1 = 0/1$ in A_x. Cross-multiplication alone would have required $y = 0$ in A, which is false. The extra factor $e = x$ in the equivalence relation gives exactly the required equality. In fact

$$A_x \simeq k[x, x^{-1}].$$

The inverse map sends x to its fraction $x/1$. The forward one sends y to zero. Geometrically we have selected the part of the x-axis away from the origin. The other axis disappears, and so do functions supported on it. In general the kernel of $A \to S^{-1}A$ consists precisely of the a such that $sa = 0$ for some $s \in S$.

Proposition 3.5 (Prime ideals after localization)**.** *Extension and contraction identify* $\operatorname{Spec} S^{-1}A$ *with the subspace of* $\operatorname{Spec} A$ *consisting of primes disjoint from* S*.*

Proof. A prime $\mathfrak{q}'$ of $S^{-1}A$ contracts to a prime $\mathfrak{q}$ of A. No $s \in S$ can lie in $\mathfrak{q}$, because its image is a unit. Conversely, if a prime $\mathfrak{q}$ avoids S, the quotient $S^{-1}(A/\mathfrak{q})$ is a nonzero domain; hence the ideal $S^{-1}\mathfrak{q}$ in $S^{-1}A$ is prime. Explicitly

$$\frac{a}{s} \in S^{-1}\mathfrak{q} \quad \Longleftrightarrow \quad a \in \mathfrak{q}.$$

For the less immediate direction, equality with a fraction having numerator in $\mathfrak{q}$ implies $tua \in \mathfrak{q}$ for some $t, u \in S$. Both avoid the prime, so $a \in \mathfrak{q}$. Extension and contraction are therefore inverse. A basic open defined by

a/s corresponds to the intersection with $D(a)$, giving the claimed subspace topology. □

There is a geometric difference between the two favourite multiplicative sets. Inverting the powers of x gives the *one* open $D(x)$. Inverting the complement of (x) gives the local ring at the generic point of the line $x = 0$. Its image consists of (0) and (x) in the affine plane. This subset is the intersection of all the opens containing (x), and need not itself be open. The words "localize at x" should always be checked against the notation A_x or $A_{(x)}$.

Exercise 3.6. Let A be a ring, $I \subseteq A$ an ideal, and $I \subseteq \mathfrak{p}$ a prime ideal. Show that the image $\mathfrak{p}/I$ of $\mathfrak{p}$ in A/I is a prime ideal and that there is a natural isomorphism

$$A_{\mathfrak{p}}/IA_{\mathfrak{p}} \cong (A/I)_{\mathfrak{p}/I}$$

Solution. Consider the map on fractions

$$A_{\mathfrak{p}} \longrightarrow (A/I)_{\mathfrak{p}/I}, \qquad \frac{a}{s} \longmapsto \frac{\bar{a}}{\bar{s}}.$$

Here $\bar{s} \notin \mathfrak{p}/I$, and every denominator on the right has a lift outside $\mathfrak{p}$. This proves surjectivity. The fraction maps to zero if and only if there exists $t \notin \mathfrak{p}$ such that $ta \in I$. In that case $a/s = (ta)/(ts) \in IA_{\mathfrak{p}}$. The reverse inclusion is immediate. In particular the residue field of $A_{\mathfrak{p}}$ is $(A/\mathfrak{p})_{(0)}$: we first impose the equations of the closure of the point, and then allow division by every nonzero element on that closure. This proves the formula $\kappa(\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p})$ used at the start. □

The last sentence of the Remark 3.4 implies that $A_{\mathfrak{p}}$ is a local ring.

Definition 3.7. A ring A is called *local* if it contains a unique maximal ideal $\mathfrak{m}$. The field $A/\mathfrak{m}$ is called *residue field* of A.

Example 3.8. If A is an integral domain, then the ideal (0) is a prime ideal and $\mathrm{Frac}(A) = A_{(0)}$ is a field, called the *field of fractions* of A.

Example 3.9. Let k be an algebraically closed field.

(1) $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} \subseteq k(x_1, \ldots, x_n)$ is the ring of rational functions whose denominator has non-zero constant term. In particular, in classical terminology, note that these are exactly the rational functions that do not have a pole at the point $(0, \ldots, 0) \in k^n$.

(2) $(k[x, y, z]/(z^2 - xy))_{(x,y,z)} \cong k[x, y, z]_{(x,y,z)}/(z^2 - xy)$ by Exercise 3.6. Thus, the elements in this ring can be thought of as equivalence classes of rational functions on k^3 which do not have a pole at the vertex of the quadratic cone.

A useful distinction in the plane is that y is invertible in $k[x, y]_{(x)}$, whereas it belongs to the maximal ideal of $k[x, y]_{(x,y)}$. The former ring examines the generic point of the line $x = 0$. Its residue field is $k(y)$; the

latter examines a closed point on the same line. The universal property of $S^{-1}A$ explains both descriptions: a map $A \to B$ taking every element of S to a unit extends uniquely, by $a/s \mapsto \varphi(a)\varphi(s)^{-1}$.

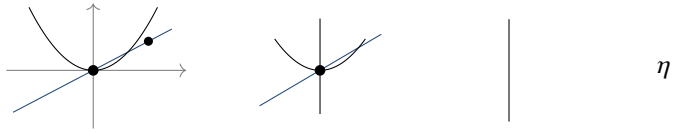


FIGURE 5. Successive localizations of the plane, drawn as in the lecture. Curves represent their generic points. At (x, y) only subvarieties through the origin survive. At (x) only the generic point of that line and the generic point of the plane survive. At (0) only the latter remains.

Exercise 3.10 (Finding a regular representative of a fraction)**.** On the quadric $A = k[x, y, z]/(xy - z^2)$ compare the fraction z/x at the generic points of $P = V(x, z)$ and $Q = V(y, z)$. What can one say at the vertex?

Solution. The denominator x lies outside (y, z), so the displayed expression gives an element of A_Q. At P, however, the element y is invertible, $x = z^2/y$, and $A_P = k(y)[z]_{(z)}$. In that ring $z/x = y/z$ has a pole of order one. At the vertex it cannot be regular: localizing the vertex ring further at (x, z) would then make it regular at P as well. A denominator which happens to vanish may be removable, so the original expression alone could not settle the question at P. The second chart settles it and anticipates our use of valuations. □

3.2. **Dimension, height and transcendence degree.** A closed subset is irreducible if it is nonempty and cannot be written as a union of two proper closed subsets. The closure of $\{\mathfrak{p}\}$ is $V(\mathfrak{p})$, which is irreducible. Conversely, if $V(I)$ is irreducible and $ab \in \sqrt{I}$, then $V(I) \subseteq V(a) \cup V(b)$ forces a or b to lie in $\sqrt{I}$. Thus $\sqrt{I}$ is prime. A chain of irreducible closed subsets is therefore a chain of primes, with the inclusions reversed.

The *height* of $\mathfrak{p}$ is the supremum of the lengths r of chains $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$. By the description of localization this is $\dim A_{\mathfrak{p}}$. The *Krull dimension* $\dim A$ is the supremum of the heights of its primes. A maximal chain and a chain of greatest possible length need not mean the same thing for an arbitrary ring.

Theorem 3.11 (Krull's height theorem)**.** *In a Noetherian ring every prime minimal over an ideal generated by r elements has height at most r. Conversely, a prime of height r is minimal over some r-generated ideal. In a domain a minimal prime over a nonzero nonunit principal ideal has height one.*

Proof. The height bound is Krull's height theorem; see [AM69, Ch. 11, Theorem 11.14 and Corollary 11.16]. For the converse choose a system of parameters in $A_{\mathfrak{p}}$ and clear the denominators. For the last assertion, the zero prime does not contain the given element. This is the precise local content of the picture that one equation cuts out a hypersurface. In particular, if $\mathfrak{m}$ is the maximal ideal of a Noetherian local ring, any generating set for $\mathfrak{m}$ has at least $\dim A$ elements [AM69, Ch. 11]. □

Corollary 3.12. *Every Noetherian local ring has finite Krull dimension. In fact* $\dim A \le \dim_{\kappa} \mathfrak{m}/\mathfrak{m}^2$ *for its residue field* κ.

Proof. The maximal ideal $\mathfrak{m}$ is finitely generated. Lifts of a basis of $\mathfrak{m}/\mathfrak{m}^2$ generate it by Nakayama's lemma (proved below). The only prime over this ideal is $\mathfrak{m}$ itself. Apply the height theorem with this generating set and use $\dim A = \operatorname{ht} \mathfrak{m}$. □

In $k[x, y]$, the primes (0), (x) and (x, y) have heights $0, 1, 2$. The chain they form gives the lower bounds and the height theorem applied to their displayed generators gives the upper bounds. Localizing at (x) removes (x, y) from the picture, so the dimension drops to one although there are still two independent rational functions in the fraction field. For a field extension K/k, algebraic independence of $t_1, \dots, t_d$ means that no nonzero polynomial over k vanishes on the tuple. It is a transcendence basis when K is algebraic over $k(t_1, \dots, t_d)$; the cardinality of a basis is the transcendence degree.

There is a second way of counting dimensions, now using the generic point.

Theorem 3.13 (Dimension and transcendence degree)**.** *For a finitely generated integral* k*-algebra* A *one has*

$$\dim A = \operatorname{trdeg}_k \operatorname{Frac}(A).$$

Proof. Noether normalization [Sta26, Tag 00OW] gives a finite integral inclusion $k[t_1, \dots, t_d] \subset A$ with the t_i algebraically independent. Lying over, going up and incomparability show that the dimensions agree. Passing to fraction fields gives a finite extension. Thus $d = \dim A = \operatorname{trdeg}_k \operatorname{Frac}(A)$. □

For $A = k[x, y, z]/(z^2 - xy)$ the subalgebra $k[x, y]$ already works. The chain $(0) \subset (x, z) \subset (x, y, z)$ realizes dimension two. At the prime (x, z) the local ring has dimension one, although its fraction field is still the function field of a surface.

Two dimension counts on the same surface. Let us compute a neighbourhood of the curve $P = V(x, z)$ on $S = V(xy - z^2)$. Passing to the quotient by (x, z) leaves $k[y]$, so P is an affine line. At its generic point y is a unit, and the presentation of the surface there becomes

$$k[x, y, z, 1/y]/(xy - z^2) = k[y, 1/y, z], \qquad x = z^2/y.$$

Localizing at (x, z) now means localizing the right-hand side at (z). Every nonzero polynomial in y becomes a unit. The result is $k(y)[z]_{(z)}$, a one-dimensional local ring with residue field $k(y)$. At the vertex $o = (x, y, z)$ the function y belongs to the maximal ideal, and the same elimination is unavailable. The chain $\{o\} \subsetneq P \subsetneq S$ exhibits two dimensions there.

One way to organize these computations for a finitely generated domain A over k is the dimension formula

$$\operatorname{ht} \mathfrak{p} + \operatorname{trdeg}_k \kappa(\mathfrak{p}) = \operatorname{trdeg}_k \operatorname{Frac} A \qquad \text{[Har77, Ch. I, Thm. 1.8A].}$$

The first term counts dimensions transverse to the closure of the point and the second counts parameters along it. At the vertex of the cone the two terms are $2 + 0$, at the generic point of P they are $1 + 1$, and at the generic point of the cone they are $0 + 2$. This formula requires a finite-type domain; there is no reason to identify the dimension of a localized ring with the transcendence degree of its fraction field over k.

Exercise 3.14 (A component can change the local dimension)**.** In $\mathbb{A}^3$ take the union of the plane $z = 0$ and the line $x = y = 0$. Compute its local dimension at the origin and at $(0, 0, 1)$.

Solution. The ideal of the union is $(z) \cap (x, y) = (xz, yz)$. At $(0, 0, 1)$ the coordinate z is a unit, so the localized ideal becomes (x, y) and only the line remains. The dimension there is one. At the origin both minimal primes survive, with quotient dimensions two and one, so the local dimension is two. No irreducible chain can start on one component and gain extra length by moving to the other: its smallest prime contains a minimal prime and the entire chain then contains that same minimal prime. The maximum over the components is the relevant number. □

We now keep the point fixed and ask which functions vanish to first order there. Comparing this new count with local dimension will allow us to detect singularities.

End of lecture 3

3.3. **Tangent spaces and regularity.** For a differentiable manifold the differential at a point kills products of two functions vanishing there. If I is the ideal of germs vanishing at the point, I/I^2 records their linear parts. The same idea makes sense without a metric, a limit or a choice of paths.

Definition 3.15. Let A be a Noetherian local ring and $\mathfrak{m} \in \operatorname{Spec} A$ its maximal ideal.

(1) The *Zariski cotangent space of* A is $\mathfrak{m}/\mathfrak{m}^2$ considered as a vector space over the field $A/\mathfrak{m}$.
(2) The *embedding dimension of* A *at* $\mathfrak{m}$ [3] is $\operatorname{edim}(A) = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

[3] We will see later in the context of Cohen's structure theorem why this is called embedding dimension

Remark 3.16. Clearly, $\operatorname{edim}(A)$ is bounded above by any number of generators of $\mathfrak{m}$, because the image of these generators in $\mathfrak{m}/\mathfrak{m}^2$ generates the Zariski cotangent space. Conversely, Nakayama's lemma implies that any basis of $\mathfrak{m}/\mathfrak{m}^2$ can be lifted to a generating set of $\mathfrak{m}$. Hence, we could have also defined the embedding dimension of a local ring A as the cardinality of a minimal generating set of its maximal ideal.

It follows from the height theorem that $\dim A \leq \operatorname{edim} A$. Equality defines a *regular local ring*. We call a point of a Noetherian scheme singular if its local ring is not regular. Its intrinsic tangent space is the $\kappa(\mathfrak{p})$-dual of $\mathfrak{p}A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})^2$. At a k-rational point of a k-variety this has a second description: a tangent vector is a k-algebra map to the dual numbers $k[\epsilon]/(\epsilon^2)$ reducing to evaluation at the point. The coefficient of ϵ is a derivation, which vanishes on $\mathfrak{m}^2$ and constants. Conversely any k-linear functional on $\mathfrak{m}/\mathfrak{m}^2$ gives such a derivation. At a nonclosed point the relative tangent space over k can also see derivations of its residue field.

Linear terms, derivations and minimal generators.

Proposition 3.17 (Tangent vectors and dual numbers)**.** *At a k-rational point the vector space of k-derivations to its residue field is dual to $\mathfrak{m}/\mathfrak{m}^2$. Its elements are equivalently the lifts of evaluation to $k[\epsilon]/(\epsilon^2)$.*

Proof. Write a tangent vector at a k-rational point as $D : A \to k$. Its Leibniz rule is

$$D(fg) = f(a)D(g) + g(a)D(f).$$

When $f, g \in \mathfrak{m}_a$, both terms on the right vanish, so every such derivation kills $\mathfrak{m}_a^2$. If $h = h(a) + (h - h(a))$, then $D(h)$ depends only on the class of $h - h(a)$ in $\mathfrak{m}_a/\mathfrak{m}_a^2$. Conversely a linear functional λ on that quotient defines

$$h \longmapsto h(a) + \epsilon\,\lambda(h - h(a)) \in k[\epsilon]/(\epsilon^2).$$

To check multiplication, expand $fg - f(a)g(a)$ and discard the product $(f - f(a))(g - g(a))$ in $\mathfrak{m}_a^2$. Thus the tangent and cotangent spaces are dual. A tangent vector is covariant in a map of varieties, as composing maps from the dual numbers shows. Linear functions on tangent vectors go in the other direction. □

Lemma 3.18 (Nakayama)**.** *For a finite module M over a local ring $(A, \mathfrak{m})$, the equality $M = \mathfrak{m}M$ implies $M = 0$. A basis of $M/\mathfrak{m}M$ lifts to a minimal generating set of M.*

Proof. Suppose M is a finite module over $(A, \mathfrak{m})$, and $M = \mathfrak{m}M$. Choose generators $e_1, \ldots, e_r$ and write $e_i = \sum a_{ij} e_j$ with all $a_{ij} \in \mathfrak{m}$. Multiplying the system by the adjugate of $I - (a_{ij})$ shows that $\det(I - (a_{ij}))$ annihilates M. Its residue is one, so it is a unit, proving $M = 0$. For the assertion on generators choose lifts v_i of a basis of $M/\mathfrak{m}M$ and let $N = \sum Av_i$. Then $M/N = \mathfrak{m}(M/N)$, so the first assertion gives $M = N$. Omitting a lift would

leave a proper subspace after reduction, proving minimality. Taking $M = \mathfrak{m}$ gives the application to the point itself. This explains why a computation modulo *squares* tells us the precise minimal number of equations for the point as an ideal in its local ring. □

Example 3.19. At a rational point of $\mathbb{A}^n_k$ the differences $x_i - a_i$ give a basis of $\mathfrak{m}/\mathfrak{m}^2$, and dimension and embedding dimension both equal n. On the quadric cone $uv = w^2$, the equation has no linear term at the origin, so the embedding dimension is three and the dimension is two. On $D(u)$ we can eliminate v, leaving $k[u, u^{-1}, w]$, and on $D(v)$ we can eliminate u. All points away from the vertex are regular; here we use that localizations of a regular local ring are regular [Sta26, Tag 00NN]. On the cusp $y^2 = x^3$, on the cubic $y^2 = x^2(1 + x)$ and on two transverse lines $xy = 0$, the origin has dimension one and embedding dimension two. These numbers detect singularity, but they do not tell these curves apart.

The cubic has two distinct tangent lines when 2 is invertible. In characteristic two the change $Y = y + x$ gives $Y^2 = x^3$ instead: the same integral equation now defines a cusp.

Example 3.20 (Computing the quotient by the square)**.** In the regular ambient ring $R = k[x, y]_{(x,y)}$ put $\mathfrak{n} = (x, y)$. For the plane curve $A = R/(f)$ at the origin, the natural map $\mathfrak{n} \to \mathfrak{m}_A$ is onto and

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \simeq \mathfrak{n}/(\mathfrak{n}^2 + (f)).$$

Thus if $f = y - x^2$, the relation reads $y = 0$ in the cotangent space; there is one basis element, x. Indeed the whole curve is isomorphic to the line. If $f = y^2 - x^3$, then f already belongs to $\mathfrak{n}^2$. There is no relation among x, y at first order. The classes span since we are reducing modulo products, and they are independent since no nonzero linear polynomial has order two. The dimension is two at that point of a one-dimensional curve. For the surface $xy = z^2$ there are instead three independent classes at the origin. These computations use only the degree-one part of the equation and remain valid in characteristic two.

The tangent cone remembers another order. For a local ring set

$$\operatorname{gr}_{\mathfrak{m}} A = (A/\mathfrak{m}) \oplus (\mathfrak{m}/\mathfrak{m}^2) \oplus (\mathfrak{m}^2/\mathfrak{m}^3) \oplus \cdots .$$

Multiplication sends degrees i, j to degree $i + j$. Each generator of $\mathfrak{m}$ contributes its class in degree one. Products of r such generators span the degree-r part, so there is a surjection $\operatorname{Sym}_\kappa(\mathfrak{m}/\mathfrak{m}^2) \to \operatorname{gr}_{\mathfrak{m}} A$. On spectra this embeds the tangent cone into the tangent space. If $A = R/(f)$ with R a polynomial local ring at the origin and $f = f_m + f_{m+1} + \cdots$, with f_m the first nonzero homogeneous piece, then

$$\operatorname{gr}_{\mathfrak{m}} A \simeq k[X_1, \ldots, X_n]/(f_m).$$

The initial form of hf is the product of the initial forms since the ambient graded ring is a domain. This proves the assertion for a principal ideal; for

a general ideal one must take initial forms of *all* its elements. Cancellation among lowest terms of given generators may produce further relations.

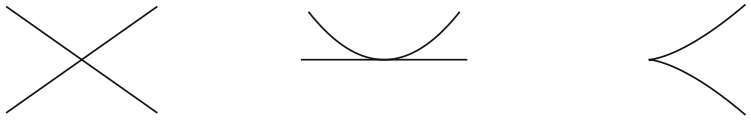
For the cusp the tangent cone is the double line $y^2 = 0$. Its tangent space is the plane. For the nodal cubic the tangent cone has two distinct lines when 2 is invertible. For the quadric cone the initial equation is its entire homogeneous equation. Thus the tautological terms "tangent space", "tangent cone" and "tangent directions" give three precise constructions: the last refers to the projectivization of the cone. Later it will occur as an exceptional fibre.

3.4. **Three ways for two arcs to meet.** Work in characteristic different from two. At the origin the cubic $y^2 = x^2(1+x)$ has tangent lines $y = \pm x$. So does the union of the lines $y^2 = x^2$. Are their local rings isomorphic? The local ring of the cubic is a domain: $1 + x$ is not a square in $k(x)$, as its zero at -1 has odd order. The local ring of the pair of lines has nonzero zero divisors $y - x$ and $y + x$. Yet a sufficiently small drawing of the two curves over the reals suggests the same crossing. We shall find the missing square root in a larger ring.



FIGURE 6. Two algebraic curves with the same formal crossing at the marked point. On the left the factors are polynomials. On the right they require a power series.

Other crossings have a different local invariant. In the local ring of $y(y - x^s) = 0$ at the origin, the quotient by the sum of its two minimal primes is $k[x]_{(x)}/(x^s)$. This has length s. The transverse pair ($s = 1$) and a tangent pair ($s = 2$) can be told apart in arbitrarily small neighbourhoods. The useful notion of equivalence must remember this length while allowing a single algebraic curve to acquire two local branches.



4. Completion and changes of formal coordinates

Exercise 4.1 (When do two smooth branches have contact s?)**.** For $s \geq 1$ let $A_s = k[[x, y]]/(y(y - x^s))$. Show directly from its ring that A_1 and A_2 cannot be isomorphic. Does the same argument work before completion?

Solution. The two minimal primes are (y) and $(y-x^s)$; they are the kernels of the two evaluations in $k[[x]]$. Their sum has quotient $k[[x]]/(x^s)$. Any ring isomorphism permutes the minimal primes, so the length of the quotient by their sum is an invariant. It equals s, separating A_1 from A_2 although their dimensions and their embedding dimensions agree. In the local polynomial ring the same proof gives $k[x]_{(x)}/(x^s)$ of the same length. The branches in this example were already present before completion. For the irreducible nodal cubic the two minimal primes appear only after completion. The two questions, whether branches exist separately and how closely they meet, are thus distinguished by the examples. □

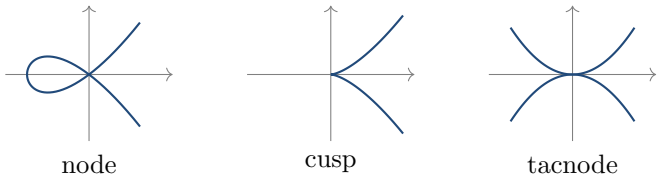


FIGURE 7. Real pictures of $y^2 = x^2(1+x)$, $y^2 = x^3$ and $y^2 = x^4$. The node has two tangent directions; the cusp and the tacnode each have one tangent direction, counted twice.

4.1. **Keeping every infinitesimal neighbourhood.** For a local ring $(A, \mathfrak{m})$ the ring $A/\mathfrak{m}^r$ contains the information visible through order $r-1$. Its spectrum has a single underlying point, but its functions distinguish jets at that point. A compatible choice at every order is a point of the inverse limit

$$\widehat{A} = \varprojlim_{r \geq 1} A/\mathfrak{m}^r = \{(a_r) : a_r \in A/\mathfrak{m}^r,\ a_{r+1} \bmod \mathfrak{m}^r = a_r\}.$$

Addition and multiplication are componentwise. There is a natural homomorphism $A \to \widehat{A}$ taking an element to all its truncations. A ring for which this is an isomorphism is called complete (for this topology). The simplest computation already gives formal power series: for $A = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, each class modulo $\mathfrak{m}^r$ has a unique polynomial representative of total degree $< r$. The inverses of denominators with nonzero constant term are given modulo $\mathfrak{m}^r$ by a finite geometric series. Compatibility means that once a coefficient occurs it is never changed, so $\widehat{A} = k[[x_1, \ldots, x_n]]$. Applying the same construction to this power series ring returns the ring itself.

Example 4.2 (Writing a formal function one layer at a time)**.** In $A = k[x,y]_{(x,y)}$, let $A_r = A/(x,y)^r$. In A_1 we see only the constant c. A lift to A_2 is $c+ax+by$. For a lift to A_3 we may freely choose the three coefficients of x^2, xy, y^2. The transition map $A_3 \to A_2$ discards precisely these three coefficients. Thus the compatibility required in the inverse limit allows infinitely many coefficients but no correction of a previously chosen coefficient. As a

concrete check, in A_4,

$$(1-x-y)^{-1} = 1 + (x+y) + (x+y)^2 + (x+y)^3.$$

The product with $1-x-y$ is $1-(x+y)^4$, hence one in A_4. Passing to the limit gives an inverse without making any claim of real or complex convergence.

There is also a universal property. If B has maps $b_r : B \to A_r$ compatible with truncation, each $b \in B$ gives the tuple $(b_r(b))_r$. Addition and multiplication are componentwise, so this defines a ring map $B \to \widehat{A}$; its composites with the projections are the prescribed b_r. This map is unique, because agreement in every component is equality of tuples. Functions are being assembled from all finite orders at once. This is different from the direct limit of rings of functions on smaller open neighbourhoods, which gave the local ring.

One may also keep the infinitesimal neighbourhoods as spaces. The formal spectrum $\operatorname{Spf} \widehat{A}$, with its $\mathfrak{m}$-adic topology, has the closed point as its underlying space and the entire compatible system of structure rings as its functions. The ordinary spectrum $\operatorname{Spec} \widehat{A}$ can have many more points: for $k[[x]]$ it also has the generic point (0). This distinction matters when we speak of studying a formal neighbourhood. Here we shall do so by computing with its complete local ring.

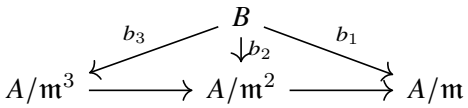


FIGURE 8. Three stages of a compatible cone. Adding all stages and imposing every reduction relation gives the unique map from B to $\widehat{A}$.

The language of adic geometry takes the idea of retaining a topology on the ring in another direction. The formal spectrum here has as points just the primes containing the ideal of definition; for the $\mathfrak{m}$-adic local ring the only such prime is $\mathfrak{m}$. In Huber's adic spaces points instead involve continuous valuations. For instance the x-adic order of a Laurent series measures its lowest exponent, the same calculation we shall meet for divisors. Perfectoid spaces belong to this latter theory and involve additional conditions on a complete nonarchimedean ring and its Frobenius [Sch12]. For the surface germ at hand the entire object we use is already accessible through the elementary sequence A_r.

End of lecture 4

4.2. **What completion preserves.** For a Noetherian local ring $(A, \mathfrak{m})$ the completion is Noetherian and local, with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{A}$. The map $A \to \widehat{A}$ is faithfully flat. For a finite A-module M, its completion is $M \otimes_A \widehat{A}$.

These facts use the Artin–Rees lemma [AM69, Ch. 10]. In particular,

$$A/\mathfrak{m}^r \simeq \widehat{A}/\widehat{\mathfrak{m}}^r, \qquad \mathfrak{m}^r/\mathfrak{m}^{r+1} \simeq \widehat{\mathfrak{m}}^r/\widehat{\mathfrak{m}}^{r+1}, \qquad \widehat{A/I} \simeq \widehat{A}/I\widehat{A}$$

for $r \geq 1$ and $I \subseteq \mathfrak{m}$. A local homomorphism gives compatible maps on truncations and hence a homomorphism on completions. The natural map into the completion is injective by Krull's intersection theorem $\bigcap_{r\geq 1} \mathfrak{m}^r = 0$. Completion preserves embedding dimension by the second isomorphism and dimension by equality of the Hilbert–Samuel functions (the degree of the eventual length polynomial is the dimension). Thus regularity passes in both directions. The example of the cubic will show why irreducibility requires more care.

Which information survives the inverse limit? There are three quite different assertions in the preceding paragraph. Injectivity tells us that two distinct elements of A can be told apart at some finite order. Exactness tells us that imposing a finite set of equations before passing to the limit has the same effect as imposing the equations in the completed ring. Neither assertion says that every element of the completed ring already belongs to A. For example, the ring $k[x]_{(x)}$ contains exactly the rational functions regular at zero. In its completion every sequence of coefficients is allowed. The series $\sum_{j\geq 2} x^{j!}$ cannot be the expansion of a rational function. For a rational series, multiplication by a polynomial with nonzero constant term gives a polynomial, so its coefficients eventually satisfy a fixed linear recurrence. Arbitrarily long strings of zeros followed by a nonzero coefficient violate any such recurrence.

Proposition 4.3 (Exactness of completion)**.** *For finite modules over a Noetherian local ring, completion is exact and the natural map of a module to its completion is injective.*

Proof. For finite modules $N \subset M$, the Artin–Rees lemma compares two ways of saying that an element of N is small. There is a c such that, for $n \geq c$,

$$N \cap \mathfrak{m}^n M = \mathfrak{m}^{n-c}(N \cap \mathfrak{m}^c M).$$

Thus the filtration induced from M and the $\mathfrak{m}$-adic filtration of N define the same topology. Apply this to $N = \bigcap_{n\geq 0} \mathfrak{m}^n M$, itself a finite module because M is Noetherian. The left side is N for every n, and $n = c+1$ yields $N = \mathfrak{m}N$. Nakayama gives $N = 0$. For exactness, start with $0 \to N \to M \to P \to 0$. At level n the kernel of $M/\mathfrak{m}^n M \to P/\mathfrak{m}^n P$ is $N/(N \cap \mathfrak{m}^n M)$. The transition maps in these kernels are surjective: a representative at one level is also a representative at the next. One can therefore lift successively and correct each lift in the kernel to make it compatible with the previous one. Passing to the limit gives an exact sequence, and Artin–Rees identifies its first term with $\widehat{N}$. Taking $N = I$, $M = A$ gives the assertion about quotient rings. Applying exactness to a finite presentation of M identifies its completion with the cokernel of the same matrix over $\widehat{A}$. □

Corollary 4.4 (Invariance of dimension and regularity)**.** *For a Noetherian local ring A one has* $\dim A = \dim \widehat{A}$ *and* $\operatorname{edim} A = \operatorname{edim} \widehat{A}$*. In particular the two rings are simultaneously regular or singular.*

Proof. The dimension assertion has an especially concrete interpretation. The number of coefficients needed to record a function up to order n is $\ell(A/\mathfrak{m}^{n+1})$. Its growth, for large n, detects $\dim A$. For a smooth plane point the monomials of total degree at most n give $(n+1)(n+2)/2$ coefficients. For the cusp $y^2 = x^3$, the initial equation is y^2, so the basis of its associated graded ring consists of $1, x, x^2, \ldots$ and $y, xy, x^2y, \ldots$. Its length function is $1 + 2n$. One sees both $\dim A = 1$ and multiplicity two in this count. Completion keeps every one of these finite lists of coefficients. It is thus unnecessary, and in general impossible, to set up a bijection between all the prime ideals before and after completion to prove equality of dimensions. □

We say that two local k-algebras are *analytically isomorphic* when their completions are isomorphic as local k-algebras. Throughout the notes this means formal equivalence. No condition of convergence of a series over $\mathbb{C}$ is imposed.

4.3. **Inverting a substitution.** A series $f \in k[[x_1, \ldots, x_n]]$ is a unit precisely when $f(0) \neq 0$. Divide by its constant term and invert $1-h$ by $\sum_{i \geq 0} h^i$: at any fixed order only finitely many summands contribute. More generally a substitution $x_i \mapsto f_i \in \mathfrak{m}$ is an automorphism if and only if the linear parts of the f_i form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Necessity follows by passing to the cotangent space. For sufficiency first undo the invertible linear substitution. Then the f_i are x_i modulo $\mathfrak{m}^2$. If an inverse has been found through degree $d-1$, its error in degree d is a tuple of homogeneous forms. Subtracting this tuple from the tentative inverse cancels the error, because substituting $f_i = x_i + O(\mathfrak{m}^2)$ in a degree-d form changes only terms of larger degree. Taking the compatible limit constructs the inverse; the same recursion on the other side gives a two-sided inverse. This is the formal inverse function theorem.

Example 4.5. The substitution $x \mapsto x + x^2$ is an automorphism of $k[[x]]$. Write its inverse as $g(y) = \sum_{i \geq 1} a_i y^i$. The equation $g + g^2 = y$ gives

$$a_1 = 1, \qquad a_n = -\sum_{i=1}^{n-1} a_i a_{n-i} \quad (n \geq 2).$$

Thus $g = y - y^2 + 2y^3 - 5y^4 + 14y^5 + \cdots$, with the integers reduced in k if necessary. The same substitution on $k[x]$ has no polynomial inverse, since the degree of a composition of nonconstant polynomials is the product of the degrees. Formal coordinates have given us just the extra inverse we need.

Two variables make the coordinate condition more visible. The substitution $(x, y) \mapsto (x + y^2, y)$ is triangular; its inverse is $(x - y^2, y)$. The substitution $(x, y) \mapsto (x^2, y)$ preserves the origin but kills the class of x in the cotangent space. No term of order two or higher in a proposed inverse can restore this lost linear coefficient. In the one-variable example above, the polynomial map $x \mapsto x + x^2$ already has the correct linear part. There is no algebraic need to choose a sign in a quadratic formula to invert it. At each order exactly one coefficient corrects the previous error. This is particularly convenient in characteristic two, where the quadratic formula with a denominator 2 would make no sense.

Exercise 4.6 (Inversion coefficient by coefficient)**.** Determine a closed formula for the inverse of $x+x^2$ and explain how to interpret it in characteristic two.

Solution. In characteristic zero the equation $g + g^2 = y$ gives $g = (\sqrt{1 + 4y} - 1)/2$, choosing constant term zero. The binomial expansion yields

$$[y^n]\, g = (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1} \quad (n \geq 1).$$

Equivalently these are the coefficients given by Lagrange inversion for $g = y/(1 + g)$. To verify the binomial form directly, expand $(1 + 4y)^{1/2}$ and cancel the even factors in $n!$. The resulting positive integers are integral because $\binom{2n-2}{n-1}/n = \binom{2n-2}{n-1} - \binom{2n-2}{n}$ (with $\binom{0}{1} = 0$). The recurrence from polynomial multiplication has integral coefficients and leading coefficient one. Hence the integer answer can first be formed and then reduced in any field. In characteristic two there is also an especially short answer: $g = y + y^2 + y^4 + y^8 + \cdots$. Squaring shifts every exponent one place in this list, giving $g + g^2 = y$. Dividing a formal expression by 2 in the coefficient field would have obscured this calculation. □

4.4. **Hensel lifting.**

Theorem 4.7 (Hensel's lemma)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring. A monic $F \in R[T]$ whose reduction is a product of pairwise coprime monic polynomials factors uniquely into monic polynomials lifting those factors, of the same degrees. In the root version, if $F(a_0) \in \mathfrak{m}$ and $F'(a_0)$ is a unit, there is a unique root $a \equiv a_0 \bmod \mathfrak{m}$. The root version also applies to a nonmonic polynomial.*

Proof. For a simple root, set $a_{j+1} = a_j - F(a_j)/F'(a_j)$. The polynomial identity $F(a + h) = F(a) + hF'(a) + h^2Q(a, h)$ shows that if the old error lies in $\mathfrak{m}^r$, the new one lies in $\mathfrak{m}^{2r}$. All derivatives encountered are units. Completeness gives the root, and $F(a)-F(b) = (a-b)(F'(b)+(a-b)c)$ gives uniqueness in its residue class. For a factorization start with two factors of degrees s, t. The linear map $(u, v) \mapsto \bar{g}v + \bar{h}u$, with $\deg u < s$, $\deg v < t$, is a bijection onto polynomials of degree $< s + t$ over the residue field:

coprimality gives injectivity, and the dimensions agree. At each successive order this map supplies the corrections, now with coefficients in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$. More factors follow by induction. □

Example 4.8 (A square root in $\mathbb{Z}_7$)**.** The congruence $T^2 = 8 \bmod 7$ has the two solutions ± 1. To lift a modulo 7^r, write the next approximation $a + 7^r b$. The digit b satisfies $2ab \equiv (8 - a^2)/7^r \bmod 7$. Starting at 1 gives $b = 4$ and $a = 29 \bmod 49$. Now $(8 - 29^2)/49 = -17$, whence $b = 2 \bmod 7$ and $a = 127 \bmod 343$. The compatible roots converge in $\mathbb{Z}_7 = \varprojlim \mathbb{Z}/7^r$; their negatives form the other solution. A derivative vanishing in the residue field would remove the uniqueness in this step.

The Newton step without an analytic Taylor formula. Divide the polynomial $F(T) - F(a)$ by the monic linear polynomial $T - a$. The quotient $G(T)$ satisfies $G(a) = F'(a)$, as one sees on a monomial by the identity $(T^n - a^n)/(T - a) = \sum_{j=0}^{n-1} T^{n-1-j} a^j$. Divide $G(T) - G(a)$ once more by $T - a$. For a polynomial H over the same ring one obtains

$$F(T) = F(a) + (T - a)F'(a) + (T - a)^2 H(T).$$

Monic division works over every commutative coefficient ring. Substituting $a - F(a)/F'(a)$ now cancels the first two terms exactly. In particular $a_{j+1} - a_j \in \mathfrak{m}^{2^j}$ when $F(a_0) \in \mathfrak{m}$. This is the Cauchy estimate for the roots themselves. It also shows why $F'(a_j)$ stays invertible: its residue never changes. In the numerical example 6 is a perfectly good starting point for the other root since $6^2 - 8 = 28 \in 7\mathbb{Z}_7$ and 12 is a unit. Modulo 49 that root is $-29 \equiv 20$, modulo 343 it is $-127 \equiv 216$. The equality $28 = 0$ holds in the residue field, while 28 in $\mathbb{Z}_7$ has order exactly one.

One more digit makes the two expansions in the notebook explicit. Starting at 127, $(8 - 127^2)/343 = -47 \equiv 2 \bmod 7$ and $2 \cdot 127 \equiv 2 \bmod 7$. Thus the next digit is 1, so the two roots modulo 7^4 are 470 and 1931. The negative branch is $6 + 2 \cdot 7 + 4 \cdot 7^2 + 5 \cdot 7^3$ at this precision. These are ordinary integer representatives of residue classes; the element in $\mathbb{Z}_7$ is their entire compatible sequence.

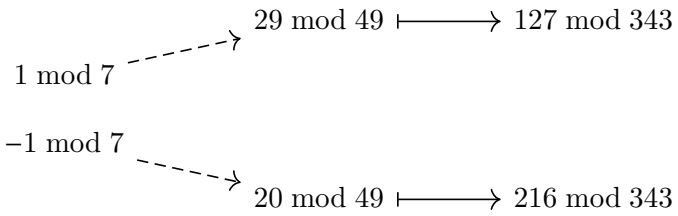
Example 4.9 (Lifting a factor, with its multiplicity)**.** Consider $F(T) = T^3 + T^2 + x$ over $k[[x]]$. In the residue field it factors as $T^2(T + 1)$. The factors are coprime even in characteristic two. We may lift the degree-one factor first. Write its root $a = -1 + s$. The equation for s becomes $s - 2s^2 + s^3 + x = 0$, whose linear coefficient at zero is one. Comparison of successive coefficients gives

$$a = -1 - x + 2x^2 - 7x^3 + O(x^4).$$

Monic division then gives the *exact* factorization

$$F(T) = (T - a)\big(T^2 + (1 + a)T + a(1 + a)\big).$$

Indeed the leading coefficients agree and the putative constant term is $-a^2(1+a) = x$. The quadratic has the prescribed reduction T^2. The lemma says nothing yet about splitting that quadratic. For example $T^2 - x$ reduces to a square, but a root in $k[[x]]$ would have half-integral x-order. In characteristic different from two, $T^2 - x^2$ has two different roots in the *same* residue class. These two examples show exactly where the coprimality in the lemma was used.



Each row follows one simple root to higher precision. Reading back down the tower means reducing modulo a lower power of seven.

For $u \in k[[x]]$ with $u(0) = 1$, apply the lemma to $T^2 - u$ at 1. If 2 is invertible there is a square root with constant term 1. More generally an sth root of a unit lifts from a chosen root of its constant term if s is invertible. In particular choose $v^2 = 1+x$. The substitution $y \mapsto vy$ transforms $y^2 - x^2(1+x)$ into $(1+x)(y^2 - x^2)$. Multiplication by this unit leaves the ideal unchanged. This is why the completed local rings of the irreducible nodal cubic and of two transverse lines are isomorphic.

The next question is what to do with an equation that is a power series in both variables. If one can make it polynomial in just one variable, the preceding tools apply again.

End of lecture 5

4.5. **Weierstrass preparation.**

Theorem 4.10 (Formal Weierstrass preparation)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, and let $f = \sum a_i t^i \in R[[t]]$. Suppose the first nonzero term of $\bar{f} \in (R/\mathfrak{m})[[t]]$ has degree s. There is a unique factorization*

$$f = u(t)(t^s + b_{s-1}t^{s-1} + \cdots + b_0)$$

with u a unit and $b_i \in \mathfrak{m}$. Moreover division by the displayed monic polynomial in $R[[t]]$ gives a unique remainder of degree $< s$ [Bou89, Ch. VII, §3].

Proof. There is a useful division argument that explains the completeness hypothesis. Write $f = t^s h + \ell$ by cutting the series after degree $s-1$. The series h is a unit and every coefficient of ℓ lies in $\mathfrak{m}$. Hence $f/h = t^s + e$ with $e \in \mathfrak{m}R[[t]]$. Every series g splits uniquely as $t^s Q(g) + P(g)$ with $P(g)$ of degree $< s$. Replace t^s in this splitting by $(f/h) - e$. The resulting error $-eQ(g)$ has coefficients in $\mathfrak{m}$. Repeating the splitting on the error gives successive errors in $\mathfrak{m}^2, \mathfrak{m}^3, \ldots$. Completeness gives a quotient and a polynomial remainder. For uniqueness reduce a relation $q(f/h) + r = 0$,

$\deg r < s$, modulo $\mathfrak{m}$, where it reads $\bar{q}t^s + \bar{r} = 0$. The same argument modulo successive powers shows $q = r = 0$.

Now divide t^s by f/h, giving $t^s = q(f/h) + r$. Modulo $\mathfrak{m}$, $q = 1$ and $r = 0$, so q is a unit and $t^s - r$ is the required monic polynomial. Any other such polynomial in (f) would differ by a polynomial of degree $< s$ in (f), hence by zero, by the uniqueness of remainders. This proves both assertions. □

The hypothesis concerns the restriction to the special fibre. The conclusion is stronger than a truncation: the equation of the entire formal hypersurface is generated by that polynomial. For $s = 1$ it gives a single graph $t = a(x)$. For instance $t^3 + t + x = 0$ determines the unique solution $t = -x + x^3 + O(x^5)$ with zero constant term. If $f = t^3 + xt + x^2$, the prepared polynomial is already f, now of degree three. Nevertheless it has a branch that one can find by setting $t = xb$. The equation for b is $xb^3 + b + 1 = 0$. Its derivative with respect to b is a unit at $(0, -1)$. Hence

$$t = a(x) = -x + x^2 - 3x^3 + O(x^4), \qquad f = (t - a)(t^2 + at + x + a^2).$$

The factor $t - a(x)$ vanishes at the origin, so it is a nonunit and cuts out a component of the formal curve. In contrast, multiplying f by $1 + x$ would give a new series whose preparation has unit factor $1 + x$ and precisely the same monic polynomial. The analogy with the analytic implicit function theorem is exact in degree one: if $F(0, 0) = 0$ and $F_t(0, 0) \neq 0$, the zero locus near the origin is a graph $t = a(x)$. The formal theorem constructs its Taylor series algebraically. At a multiple root, the polynomial factor continues to describe all nearby branches even when there is no single graph.
Projection and the meaning of the distinguished polynomial.

Corollary 4.11 (Finite projection of a formal plane curve)**.** *If $f(0, t)$ has order s in $k[[t]]$, then $k[[x, t]]/(f)$ is free of rank s over $k[[x]]$.*

Proof. Let $R = k[[x]]$. Identify the formal t-axis with the fibre $x = 0$. If the first nonzero term of $f(0, t)$ is of degree s, then the intersection of the curve with this axis has length s:

$$\dim_k k[[x, t]]/(x, f) = \dim_k k[[t]]/(t^s) = s.$$

Weierstrass division says much more. Projection onto the x-axis gives a finite free $k[[x]]$-algebra of rank s, with basis $1, t, \ldots, t^{s-1}$. Spanning is division with remainder, independence is uniqueness of the remainder. Thus intersection with the special fibre predicts how many coefficients we must keep along *every* infinitesimal neighbourhood of that fibre. A badly chosen axis can give a larger number or even be a component. For $y^2 = x^3$, projection to x has degree two, while projection to y has degree three. Taking a line outside the finite set of tangent directions makes the degree of the projection equal to the multiplicity at the origin. □

The polynomial t^3+t^2+x provides a preparation in which the unit matters. Over $k[[x]][t]$ we separated a linear factor $t - a$ and a quadratic above. In

$k[[x,t]]$ the constant coefficient of $t-a$ is 1 at the closed point, so this entire linear factor is a unit there. The quadratic is distinguished because $1+a$ and $a(1+a)$ lie in (x). It alone describes the germ at $(0,0)$. At $(0,-1)$ the situation is reversed: the quadratic is a unit and the linear factor gives the graph of a single branch. Thus the degree of a polynomial on the entire affine line and the degree of its preparation at a specified point answer different geometric questions.

Another way to use degree-one preparation is as an implicit function theorem with parameters. If $f(0,0)=0$ and $f_t(0,0)$ is invertible, the coefficient of x^n in $f(x,a(x))$ is a known expression in the earlier coefficients of a, plus $f_t(0,0)a_n$. Each equation determines exactly one a_n. The factorization $f=(t-a)u$ says that *every* series vanishing on this graph belongs to $(t-a)$. For a series g, its remainder on division by $t-a$ is simply $g(x,a(x))$. This last equality is often the fastest way to recognize the kernel of a parametrization.

5. Hypersurfaces and plane curve double points

5.1. How many variables does a singularity need?

Proposition 5.1 (Minimal presentation)**.** *An equicharacteristic complete Noetherian local algebra with given coefficient field k and embedding dimension e has a presentation by e variables with no linear relations. If it is regular, this presentation is an isomorphism.*

Proof. Let $(A,\mathfrak{m})$ be a complete Noetherian local k-algebra inducing $k\simeq A/\mathfrak{m}$. Choose $e=\operatorname{edim}A$ elements lifting a basis of $\mathfrak{m}/\mathfrak{m}^2$. Then

$$k[[X_1,\dots,X_e]]\twoheadrightarrow A,\qquad X_i\longmapsto x_i,\qquad \ker\subseteq(X_1,\dots,X_e)^2.$$

Indeed the lifts generate $\mathfrak{m}$ by Nakayama. Write each element as its constant term plus a linear combination of the x_i, then expand the coefficients and repeat. Completeness makes the resulting series an actual preimage. A linear relation in the kernel would contradict the choice of basis. This is the equicharacteristic presentation in Cohen's structure theorem; in general the existence of a coefficient field is itself part of the theorem [Sta26, Tag 032A].

In the complete equicharacteristic regular case the displayed surjection is an isomorphism: its source and target have dimension e, and any nonzero ideal of the power series domain lowers dimension. A complete local domain of dimension $e-1$ in this presentation is a hypersurface, because the kernel is a height-one prime in the factorial regular local ring $k[[X_1,\dots,X_e]]$. The domain assumption matters: $k[[x,y]]/(x^2,xy)$ has dimension one and embedding dimension two but its defining ideal needs two generators. □

Lemma 5.2 (Why a regular local ring has no zero divisors)**.** *Let $(R,\mathfrak{n})$ be Noetherian and regular with residue field k_0 and regular parameters $x_1,\dots,x_e$. Then every product of two nonzero elements of R is nonzero.*

Proof. By the characterization of regular local rings in [Sta26, Tag 00NN], $\operatorname{gr}_{\mathfrak{n}} R = k_0[\bar{x}_1, \dots, \bar{x}_e]$. Here the bar denotes initial form in degree one. For $a \neq 0$ let i be the greatest integer with $a \in \mathfrak{n}^i$, which exists by Krull intersection; write a^* for its nonzero class in degree i. If $b \neq 0$ has order j, the product a^*b^* in the polynomial ring is nonzero and is the degree-$i+j$ class of ab. In particular $ab \notin \mathfrak{n}^{i+j+1}$. This proves the assertion without assuming R complete or that its residue field embeds in R. □

For a hypersurface $A = k[[x_1, \dots, x_e]]/(f)$ with $0 \neq f \in \mathfrak{m}^2$, the *multiplicity* is the order of f, the least total degree of a term with nonzero coefficient. This agrees with the Hilbert–Samuel multiplicity since the associated graded ring is the polynomial ring modulo the first homogeneous part of f. Any isomorphism between two such minimal presentations lifts to a change of formal coordinates: lift the images of the variables and use invertibility on $\mathfrak{m}/\mathfrak{m}^2$. The two equations then differ by a unit. This equivalence is called *contact equivalence*. The initial forms transform by the linear part of the substitution and a nonzero scalar.

Minimal equations and embedded components. For the quotient by (x^2, xy) let $\mathfrak{n} = (x, y)$ in the power series ring and $I = (x^2, xy)$. Its radical is (x), so the support is just the y-axis. The class of x nevertheless survives and is killed by the maximal ideal. It records an embedded point at the origin. Since I starts in degree two, the distinct degree-two monomials x^2 and xy remain linearly independent in $I/\mathfrak{n}I$. Nakayama now says that two equations are necessary. This explains the use of the prime kernel in the hypersurface assertion: knowing the dimension of the support does not count the infinitesimal equations.

For a principal ideal (f) write $f = f_m + f_{m+1} + \cdots$ in homogeneous pieces with $f_m \neq 0$. If g has first term g_j, then gf has first term $g_j f_m$ because the graded polynomial ring is a domain. Hence the initial ideal of (f) is exactly (f_m). In two variables the Hilbert series of the tangent cone is

$$\frac{1 - z^m}{(1 - z)^2} = \frac{1 + z + \cdots + z^{m-1}}{1 - z}.$$

Every sufficiently large homogeneous piece has dimension m. Summing the dimensions gives a linear length polynomial with leading coefficient m, which proves the multiplicity assertion for a plane curve. If a variable in an equation occurs with a nonzero linear coefficient, it can instead be eliminated by the implicit function theorem. A minimal singular plane equation thus has no constant or linear term.

5.2. **Classification in multiplicity two.**

Theorem 5.3. *Let k be algebraically closed, and let $f \in k[[x, y]]$ have order two. Up to a change of coordinates and a unit the possibilities are as follows.*

(1) If f is reducible, it is y^2 or $y(y + x^s)$, $s \geq 1$.

(2) If f is irreducible and $\operatorname{char} k \neq 2$*, it is* $y^2 + x^{2m+1}$*, $m \geq 1$.*
(3) If f is irreducible and $\operatorname{char} k = 2$*, it is* $y^2 + x^{2m+1}$ *or*

$$y^2 + x^{2m-r}y + x^{2m+1}, \qquad m \geq 2, \quad 1 \leq r \leq m-1.$$

Proof. If $f = f_1 f_2$ with nonunit factors, both have order one. Choose $y = f_1$. A second copy of the same branch gives y^2 up to a unit. For distinct branches choose x so its zero line is transverse to both. The second smooth branch has equation $y = h(x)$ by formal preparation of degree one, after choosing x as parameter along each branch. The nonzero h equals $x^s v(x)$ for a unit v. Rescale the y coordinate by that unit and change sign to obtain $y(y + x^s)$. Notice that this argument does not require taking an sth root in characteristic dividing s.

For an irreducible equation choose the y-axis away from the tangent directions so that $f(0, y)$ has order two. Preparation gives $f = y^2 + a(x)y + b(x)$ up to a unit, with $\operatorname{ord}_x a \geq 1$ and $\operatorname{ord}_x b \geq 2$. If two is invertible, complete the square and write the remaining nonzero constant coefficient as $x^d u(x)$. Take a square root of the unit and rescale y. If d is even, the quadratic splits over k, so $d = 2m + 1 \geq 3$.

Suppose now that the characteristic is two. If $a = 0$, all the even-degree terms of b form a square in $k[[x]]$, since k is perfect. Remove them by a translation of y. If nothing remains, the equation is a square. Otherwise $b = x^d u$ with d odd and u a unit. Replacing x by $xu^{1/d}$ gives the pure form. This root exists by Hensel.

If $a \neq 0$, rescale y by a unit and divide the equation by its square to obtain $y^2 + x^n y + b(x)$, $n \geq 1$. If $\operatorname{ord} b \geq 2n$ (including $b = 0$), substitute $y = x^n T$ and divide by x^{2n}. The resulting $T^2 + T + b/x^{2n}$ has two simple residue roots. Their lifts give two factors of the original quadratic. Thus $\operatorname{ord} b = d < 2n$. While d is even, translate y by $cx^{d/2}$ where c^2 is the leading coefficient of b. This cancels that term and adds one of degree $n + d/2 > d$. The order strictly increases until it is odd or reaches $2n$. The latter case is reducible, so we have reached $b = x^d u$, d odd and $d < 2n$.

Set $v = u^{-1/(2n-d)}$, which exists since $2n-d$ is odd; use the new coordinates $X = vx$, $Y = v^n y$. Multiplying the old equation by v^{2n} gives $Y^2 + X^n Y + X^d$. There is one boundary case worth doing explicitly. If $n \geq d - 1$, put $X = x + x^{n-d+1}y$. Then $x^d + x^n y = x^{d-1}X$. As $d - 1$ is even, $x^{d-1} - X^{d-1}$ is divisible by y^2. Thus the equation is $X^d + Uy^2$ with U a unit (the correction to 1 is divisible by X). After division by U absorb U^{-1} into X^d by taking a dth root. These substitutions have invertible linear parts. In particular $n = d$ does *not* force reducibility. In the remaining case $d = 2m + 1$ and $m + 1 \leq n \leq 2m - 1$, giving the asserted range with $n = 2m - r$.

Finally the displayed odd-order quadratics are irreducible: if a root had integral x-order s, then among $2s, n + s, d$ the minimum would have to occur twice. With d odd and $2n > d$ this is impossible. The same order argument works for the pure form. □

Example 5.4 (Following the characteristic-two reduction)**.** The translation used in the proof can create a coefficient which did not occur in the original polynomial. Take

$$f = y^2 + x^5 y + x^4 \quad \text{in characteristic two.}$$

The substitution $y = Y + x^2$ contributes x^4 from the square and x^7 from the mixed term. The equation becomes $Y^2 + x^5 Y + x^7$: here $d = 7$, $n = 5$, $m = 3$, $r = 1$. There is just one tangent direction. One cannot split off a smooth graph by Hensel at this point because the special fibre has a double root. If there were a branch $Y = b(x)$ with $s = \operatorname{ord}_x b$, then either $s \leq 3$ and $2s$ is strictly smallest, or $s \geq 4$ and 7 is strictly smallest. A sum with a unique least-order term cannot vanish. Monic division and preparation show that a factorization would indeed give such a graph.

Compare this with $y^2 + x^4 y + x^6 + x^7 + x^9$. Translating by x^3 cancels *both* x^6 and x^7, leaving $Y^2 + x^4 Y + x^9$. Substitute $Y = x^4 T$. After division by x^8 one gets $T^2 + T + x = 0$. The roots modulo x are $0, 1$ and both derivatives equal one. If b lifts the first root, then $b + 1$ lifts the other. The original two branches are therefore the graphs $y = x^3 + x^4 b$ and $y = x^3 + x^4(b + 1)$; their intersection has length four. In the first example the cancellation exposed an odd term *below* $2n$; in the second it moved the equation past $2n$. Stopping the reduction after observing the initial even power would give the wrong answer in both examples.

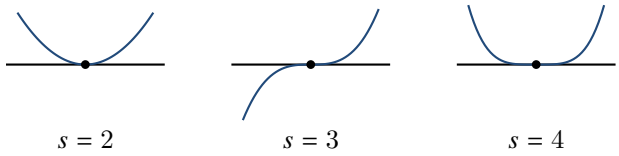


FIGURE 9. The branches $y = 0$ and $y = x^s$ drawn over the reals. Crossing or remaining on the same side detects parity in the picture; the algebraic intersection length detects the full integer s.

5.3. **Distinguishing the normal forms.** The double line is nonreduced. A reduced pair of branches is determined in the list by the length s of the intersection, computed above. For an irreducible quadratic of odd exponent $d = 2m + 1$ the normalization is a complete discrete valuation ring $k[[t]]$ (see Section 7). Normalize its valuation to have value group $\mathbb{Z}$. Since the ring is free of rank two over $k[[x]]$, $\operatorname{ord}_t x = 2$: the normalization also has rank two and the quotient by x has length two. Comparing orders in the equation gives $\operatorname{ord}_t y = d$. Every element has the unique form $u(x) + v(x)y$ and the two leading valuations have opposite parity. Its value semigroup is therefore $\langle 2, d \rangle$, which recovers d as its least odd member. Equivalently the missing values $1, 3, \ldots, 2m - 1$ give the δ-invariant m, the length of the normalization modulo the original ring.

For fixed d in characteristic two consider the Tjurina algebra

$$T_f = k[[x, y]]/(f, f_x, f_y).$$

The chain rule shows that its isomorphism class is invariant under contact equivalence. The pure equation gives length $2(d-1)$. In the mixed case $f_y = x^n$ and $f_x = nx^{n-1}y + x^{d-1}$. Modulo x^n the equation becomes $y^2 = 0$. The length is $2n$ for even n and $2n-1$ for odd n. These lengths strictly increase with n and are all smaller than the pure length. Thus the normal forms really are pairwise distinct.

For example take $d = 7$. The three allowed possibilities are the pure form and the two mixed forms with $n = 4, 5$. In the first mixed case the Jacobian relations reduce to $x^4 = 0$ and $y^2 = 0$ (the term x^6 is already zero). The basis consists of $x^i, x^i y$ for $0 \le i < 4$, giving length eight. In the second there are ten such candidates with $0 \le i < 5$, but the relation $x^4 y = 0$ removes the last one, giving length nine. In the pure case the surviving powers range from 0 to 5 in both rows, giving length twelve. All three branches have the same value semigroup $\langle 2, 7\rangle$. This is a particularly small instance in which the orders of all functions along a branch still leave a choice of analytic type.

For the broader question of which hypersurface singularities are simple for contact equivalence in positive characteristic, see Greuel and Kröning [GK90].

End of lecture 6

5.4. **Four lines and a modulus.** The answer for double points is a countable list in every characteristic, even over an uncountable field. Already four lines through the origin behave differently. Consider

$$f_\lambda = xy(x+y)(x+\lambda y) = x^3y + (1+\lambda)x^2y^2 + \lambda xy^3, \qquad \lambda \in k \setminus \{0, 1\}.$$

If two completed local rings are isomorphic, the linear part of a lift must take the four tangent lines of one equation to those of the other. Conversely that linear change suffices here because the equations are homogeneous. The directions are four points on $\mathbb{P}^1$. A projectivity is determined by the images of three ordered points, leaving the fourth as a cross-ratio. Permuting the order replaces λ, in a suitable convention, by one of

$$\lambda, \quad 1-\lambda, \quad 1/\lambda, \quad 1/(1-\lambda), \quad (\lambda-1)/\lambda, \quad \lambda/(\lambda-1).$$

Some entries may coincide. The important conclusion is that a single isomorphism class contains at most six parameters, so there are infinitely many classes, and uncountably many over an uncountable k. Blowing up the origin will put these very four points on an actual projective line.

Exercise 5.5 (A modulus already at multiplicity three)**.** Over an algebraically closed field k, consider the three smooth branches of

$$C_\lambda : \quad y(y-x^2)(y-\lambda x^2) = 0, \qquad \lambda \in k \setminus \{0, 1\}.$$

Can their common tangent and their three pairwise intersection multiplicities determine the completed local k-algebra? What happens in characteristic three?

Solution. The tangent cone is $y^3 = 0$. The intersection of the branches $y = cx^2$ and $y = c'x^2$ has ring $k[[x]]/((c - c')x^2)$ and length two. These data are independent of λ. A k-isomorphism of the completed curve rings lifts to a formal change of plane coordinates, by the minimal presentation proved above. Its linear part must preserve $y = 0$. Write

$$X = ax + by + R, \qquad Y = dy + ex^2 + S, \qquad ad \neq 0.$$

Here $R \in (x, y)^2$ and $S \in (xy, y^2) + (x, y)^3$. On the branch $y = cx^2$ this gives $X = ax + O(x^2)$ and $Y = (dc + e)x^2 + O(x^3)$. Thus the coefficient of X^2 is $(dc+e)/a^2$. One affine transformation must carry the entire unordered triple $\{0, 1, \lambda\}$ to $\{0, 1, \lambda'\}$. Conversely if $c \mapsto mc+n$ carries one triple to the other, take $X = x$ and $Y = my + nx^2$. This maps the branches exactly. Ordering the source triple gives at most six ways to normalize its first two entries to $0, 1$. Each analytic class therefore accounts for at most six values of the parameter. Over an uncountable k there are uncountably many classes, also in characteristic three. No division by three entered the argument.

There is a way to see the coefficient c. In the chart $t = y/x$ all three branches still meet at $(0, 0)$, now with equations $t = cx$. Blow up that point and set $v = t/x$. The three coefficients become three points $v = c$ on the new exceptional line. Its intersection with the old exceptional curve supplies the distinguished fourth direction $v = \infty$. A projectivity which preserves that direction acts on the other three by an affine transformation, exactly as in the calculation above. □

For a reduced curve germ over an algebraically closed field, its δ-invariant is the length of the quotient of its normalization by its local ring. The four-line example shows what this number can and cannot detect. For two smooth branches of contact s one has the exact sequence

$$0 \longrightarrow k[[x, y]]/(y(y - x^s)) \longrightarrow k[[x]] \oplus k[[x]] \xrightarrow{(a,b)\mapsto a-b} k[[x]]/(x^s) \longrightarrow 0.$$

Indeed the first map evaluates on the two branches. Its image consists of pairs congruent modulo x^s, since $a(x) + y(b(x) - a(x))/x^s$ interpolates any such pair. Thus $\delta = s$. More generally for two reduced curves with no common component the same sequence, using their two ideals, gives $\delta(C \cup D) = \delta(C) + \delta(D) + I_0(C, D)$, where I_0 is the length of the quotient by the sum of their ideals. For four distinct lines this gives $\delta = \binom{4}{2} = 6$ independently of λ. It is the relative position of the four points of the exceptional line that this length cannot detect.

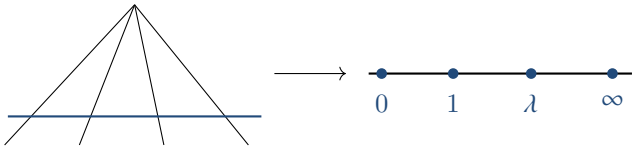


FIGURE 10. Four lines through a common point determine four points on its line of directions. Choosing three as $0, 1, \infty$ makes the remaining coordinate a cross-ratio.

Proposition 5.6 (What a double point costs in genus)**.** *Let C be a reduced projective plane curve of degree d over an algebraically closed field. Let $\widetilde{C}_i$ be the smooth projective normalizations of its h irreducible components. Then*

$$\frac{(d-1)(d-2)}{2} = \sum_{i=1}^{h} g(\widetilde{C}_i) + \sum_{p \in C} \delta_p - h + 1.$$

Proof. For the finite normalization ν the inclusion $\mathcal{O}_C \hookrightarrow \nu_* \mathcal{O}_{\widetilde{C}}$ is an isomorphism on the smooth locus. Its quotient is supported on finitely many points, and has length δ_p at p, by the definition of δ. Additivity of Euler characteristics gives $\chi(\mathcal{O}_C) = \sum_i (1 - g(\widetilde{C}_i)) - \sum_p \delta_p$. On the other hand the single degree-d equation gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

From $\chi(\mathcal{O}_{\mathbb{P}^2}(n)) = (n+2)(n+1)/2$, valid for every integer n, one obtains $1 - \chi(\mathcal{O}_C) = (d-1)(d-2)/2$. For the cohomology of the twisting sheaves used here see [Har77, Ch. III, Theorem 5.1]. □

Exercise 5.7. Compute the geometric genus of a cubic with a single ordinary node, and compare it with a cubic having a single ordinary cusp. How could one distinguish the two defects locally?

Solution. Both cubics are assumed integral. Each has arithmetic genus one. At a node the normalization allows two independent values above the point, and equality of these values imposes one condition. At a cusp the normalization $k[[t]]$ supplies the missing coefficient of t in $k[[t^2, t^3]]$. Thus $\delta = 1$ in both cases and the normalization has genus zero. The number r of branches is two for the node and one for the cusp. Over $\mathbb{C}$ one can also take the Milnor number $\mu = \dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(f_x, f_y)$. For xy the quotient is $\mathbb{C}$; for $y^2 - x^3$ it has basis $1, x$. These agree with the general formula $\mu = 2\delta - r + 1$ for a reduced complex plane curve germ [Wal04, Ch. 6]. In positive characteristic one must check the partial derivatives: in characteristic three the partial with respect to x of $y^2 - x^3$ vanishes, and the Jacobian quotient is infinite-dimensional. □

Proposition 5.8 (Multiplicities at infinitely near points)**.** *Let C' be the strict transform of a reduced plane curve over an algebraically closed field*

after blowing up a closed point p of multiplicity m. Then

$$\delta_p(C) = \frac{m(m-1)}{2} + \sum_{q \in C',\, q \mapsto p} \delta_q(C').$$

For two curves C, D without a common component, the corresponding subtraction in local intersection multiplicity is $m_p(C)m_p(D)$.

Proof. Work in a smooth projective surface containing the curve, taking projective closures for plane curves. Write π for the blow-up and E for the exceptional line. Then $C' = \pi^*C - mE$, $K' = \pi^*K + E$, $E^2 = -1$, and a pulled-back divisor has degree zero on E. In the adjunction formula $2p_a(C') - 2 = C' \cdot (C' + K')$ these identities give

$$C' \cdot (C' + K') = C \cdot (C + K) - m(m-1).$$

Normalization is unchanged by this birational modification, as are all local terms away from p. The preceding Euler characteristic argument works for a reduced curve on either surface and gives the first formula. For the second, expand $(\pi^*C - mE)(\pi^*D - nE) = C \cdot D - mn$ and remove the intersections away from p. These intersection identities are proved in [Har77, Ch. V, §§1,3]. □

Consequently δ can be read by adding $\binom{m_q}{2}$ at all successive centres until the branches are smooth and separated. If the process continues through a smooth point to achieve transversality, its multiplicity is one and the sum stays the same. For the two branches $y = 0$ and $y = x^s$, the x-chart replaces their contact exponent s by $s - 1$. Each common centre contributes $1 \cdot 1$ to their intersection number; after s centres the branches occupy different points. For a single branch $x = t^2$, $y = t^{2m+1}$, a blow-up replaces y by $y/x = t^{2m-1}$. There are m multiplicity-two centres, giving $\delta = m$. The conductor of its value semigroup is $2m$: every integer from $2m$ on occurs, and $2m - 1$ is the last gap. Dividing one parameter by another in successive charts is the geometric origin of the characteristic exponents of a branch. Puiseux expansions in characteristic zero and Hamburger–Noether expansions in arbitrary characteristic encode this sequence; see [Wal04, Chs. 2–3] for the complex case and [FH10] for plane curve resolution.

Proposition 5.9 (From characteristic exponents to the genus defect)**.** *For a complex plane branch write $\bar\beta_0 = m, \bar\beta_1, \dots, \bar\beta_g$ for the minimal generators of its value semigroup in characteristic order. Put $e_i = \gcd(\bar\beta_0, \dots, \bar\beta_i)$ and $n_i = e_{i-1}/e_i$. Then the conductor exponent c and the colength δ satisfy*

$$2\delta = c = \sum_{i=1}^{g} (n_i - 1)\bar\beta_i - m + 1.$$

Proof. For a singular branch, if the Puiseux characteristic is $(m; \beta_1, \dots, \beta_g)$, the conversion to semigroup generators starts with $\bar\beta_1 = \beta_1$ and continues by $\bar\beta_{i+1} = n_i\bar\beta_i + \beta_{i+1} - \beta_i$. The plane-branch semigroup theorem also gives

$n_i\bar\beta_i \in \langle\bar\beta_0,\dots,\bar\beta_{i-1}\rangle$; see [Tei07, §8, Proposition 8.1 and the discussion following it]. Hence every value has the form

$$qm + \sum_{i=1}^{g} a_i\bar\beta_i, \qquad q \ge 0, \quad 0 \le a_i < n_i.$$

Indeed, reduce the largest index first using that semigroup relation. All the sums with $q = 0$ have distinct residues modulo m. If two agreed modulo m, reduction modulo e_{g-1} would give $n_g \mid (a_g - a'_g)$ and therefore $a_g = a'_g$. Descending the indices gives equality of all the digits. There are $\prod_i n_i = m$ choices because $e_g = 1$. Thus each residue class has precisely one least value w_r.

The largest gap is $\max_r w_r - m$, giving the asserted conductor. The number of missing nonnegative integers in class r, for $0 \le r < m$, is $(w_r - r)/m$. In the average over all digit tuples the digit a_i takes mean value $(n_i - 1)/2$. Summing over the residues, the number of gaps is therefore $\big(\sum(n_i - 1)\bar\beta_i - m + 1\big)/2$. Finally the normalization is $\mathbb{C}[[t]]$. Filter it and the branch ring by order in t. Their associated graded pieces differ by one dimension exactly at a gap. Beyond the conductor they agree, and successive cancellation converges in the complete branch ring. This identifies the number of gaps with δ. For the smooth branch the semigroup is $\mathbb{N}_0$ and the empty sum gives $c = 0$. □

Exercise 5.10 (A branch with two characteristic exponents)**.** Over $\mathbb{C}$ take the formal branch $x = t^4,\ y = t^6 + t^7$. Calculate its value semigroup, conductor and δ. Where does an odd value first appear?

Solution. Put $R = \mathbb{C}[[x]]$ and $h = y^2 - x^3$. Substitution gives $h = 2t^{13} + t^{14}$. The four elements $1, y, h, yh$ have orders $0, 6, 13, 19$, all distinct modulo four. They form an R-basis of the branch ring. Indeed the product of $Y - y(\zeta t)$ over the four fourth roots of unity is monic of degree four with coefficients in R, and reduces to Y^4 modulo x. The four series $y(\zeta t)$ are all distinct since $\gcd(4, 6, 7) = 1$. Thus this is the minimal polynomial over $\mathbb{C}((x))$. Weierstrass division gives the basis $1, y, y^2, y^3$ in the completed ring, and replacing the last two elements by h, yh is a triangular change of basis.

Coefficients in R raise an order by a multiple of four. The four residue classes prevent cancellation between the leading terms of the four summands. The value set is therefore

$$(0 + 4\mathbb{N}_0) \ \cup\ (6 + 4\mathbb{N}_0) \ \cup\ (13 + 4\mathbb{N}_0) \ \cup\ (19 + 4\mathbb{N}_0) = \langle 4, 6, 13\rangle.$$

The gaps are $1, 2, 3, 5, 7, 9, 11, 15$. The conductor exponent is 16 and $\delta = 8$. To see why gaps count the colength, successively cancel the lowest non-gap term of a series in the normalization $\mathbb{C}[[t]]$ using an element of the branch ring of that order. Completeness makes this cancellation converge: in each of the four basis positions the x-orders tend to infinity. The missing monomials give independent residue classes since a nonzero combination has gap

order. Thus $t^{16}\mathbb{C}[[t]]$ lies in the branch ring and t^{15} does not. In particular 7 is a characteristic exponent of the parametrization but is not itself a semigroup value. For the general relation between characteristic exponents, the value semigroup and the resolution of a complex plane branch, see [Wal04, Chs. 2–4]. □

6. Graded rings, projective space and the blow-up

6.1. **How to make directions into points.** We would like to replace the origin of $\mathbb{A}^n$ by the directions of approach to it. A nonzero vector determines the same direction as any nonzero multiple. Why not just take the scalar invariants in $k[y_1, \dots, y_n]$, as we did for a finite group? Over an infinite field the only such polynomials are constants: scaling a monomial of degree d multiplies it by λ^d. We should instead look at ratios of homogeneous polynomials of equal degree. The denominator need only stay nonzero on the chart where the ratio is used.

Definition 6.1. A nonnegatively graded ring $S = \bigoplus_{d \geq 0} S_d$ has $S_d S_e \subseteq S_{d+e}$. An element of a single summand is homogeneous; an ideal is homogeneous if generated by homogeneous elements. The irrelevant ideal is $S_+ = \bigoplus_{d>0} S_d$. The points of $\operatorname{Proj} S$ are the homogeneous prime ideals not containing S_+. The closed sets are $V_+(J) = \{\mathfrak{p} \in \operatorname{Proj} S : J \subseteq \mathfrak{p}\}$ for homogeneous J. For homogeneous f of positive degree put $D_+(f) = \operatorname{Proj} S \setminus V_+(f)$.

A homogeneous quotient gives $\operatorname{Proj}(S/J) \simeq V_+(J)$ with its quotient scheme structure. The basic open $D_+(f)$ has coordinate ring $S_{(f)} = (S_f)_0$, the degree-zero part of homogeneous localization. A fraction a/f^r lies there when $\deg a = r \deg f$. On overlaps one localizes again. On degree-one charts this is just the familiar operation of dividing all coordinates by the chosen nonzero one [Har77, Ch. II, §2].

Example 6.2. Projective n-space over any ring A is $\operatorname{Proj} A[X_0, \dots, X_n]$, with the X_i of degree one. Its chart $D_+(X_i)$ is the spectrum of the polynomial ring in the n ratios X_j/X_i for $j \neq i$. On $\mathbb{P}^2$, for example, the coordinates $s = X_1/X_0, t = X_2/X_0$ become $1/s, t/s$ on the next chart. Over $k = \bar{k}$ the closed points are lines through the origin in k^{n+1}. The homogeneous prime for the line through $(a_0, \dots, a_n) \neq 0$ is the kernel of $X_i \mapsto a_i T$ into $k[T]$. This description also explains why the irrelevant ideal is omitted. A homogeneous equation defines a locus on projective space because its vanishing is independent of the chosen representative.

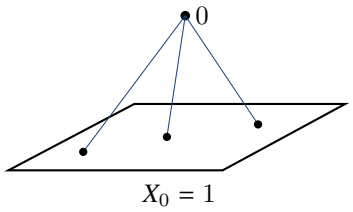


FIGURE 11. A direction in k^3 for which $X_0 \neq 0$ meets the slice $X_0 = 1$ at a unique point. The missing directions constitute the projective line $X_0 = 0$.

Lemma 6.3 (Why the degree-zero part is needed)**.** *For $S = k[X_0, \dots, X_n]$, restriction to $X_i \neq 0$ identifies $(S_{X_i})_0$ with $k[u_j : j \neq i]$, where $u_j = X_j/X_i$. On $X_j \neq 0$ in that chart one must invert u_j.*

Proof. Write a degree-zero fraction with common denominator X_i^a. Each homogeneous monomial of degree a divided by X_i^a is a monomial in the u_j, so they generate. A polynomial relation among them becomes a relation in the independent variables X_j after substituting $X_i = 1$, hence is zero. The equality $X_j^a/X_i^a = u_j^a$ gives the indicated second localization. Using all homogeneous degrees would retain an extra invertible scale coordinate. Taking degree zero identifies representatives of one direction. □

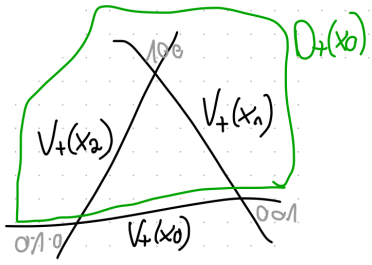


FIGURE 12. The coordinate triangle in $\mathbb{P}^2$ from the seventh lecture. On the chart $X_0 \neq 0$ the two remaining coordinate lines become the two affine axes. The omitted side $X_0 = 0$ consists of the directions at infinity.

Example 6.4 (A curve in two projective presentations)**.** Give $B = \bigoplus_{r \geq 0} k[s,t]_{3r}$ the grading in which cubic monomials have degree one. On $D_+(s^3)$ the degree-zero ring is $k[t/s]$; indeed $s^2t/s^3 = t/s$ and every other quotient is a polynomial in this ratio. On $D_+(t^3)$ it is $k[s/t]$. These opens cover: a homogeneous prime containing s^3 and t^3 contains all four cubic monomials by taking powers, hence contains the irrelevant ideal. The two coordinates are inverses on the overlap. Thus $\operatorname{Proj} B = \mathbb{P}^1$, embedded in $\mathbb{P}^3$ by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Its affine cone will give a canonical class of order three. Taking Proj kept the same curve, while choosing different homogeneous coordinates changed the cone.

Exercise 6.5. Describe the generic point of $\mathbb{P}^1_k$ and a closed point of $\mathbb{P}^1_{\mathbb{R}}$ which cannot be written with two real homogeneous coordinates.

Solution. The homogeneous prime (0) belongs to every nonempty basic open of $\operatorname{Proj} k[S,T]$ and specializes to every point. On $S \neq 0$ its residue field is $k(T/S)$. Over $\mathbb{R}$ the homogeneous prime $(S^2 + T^2)$ gives the maximal ideal $(1 + u^2)$ on the same affine chart. Its residue field is $\mathbb{C}$, and it splits

into two conjugate points over $\mathbb{C}$. Thus the homogeneous-prime definition automatically includes closed points with larger residue fields. □

6.2. **The incidence variety and the Rees algebra.** Write $y_1, \dots, y_n$ for coordinates on $\mathbb{A}^n$ and $[U_1 : \cdots : U_n]$ for those on $\mathbb{P}^{n-1}$. In their product take the equations

$$y_i U_j = y_j U_i \qquad (1 \le i, j \le n).$$

Away from the origin these determine the direction uniquely. Over the origin they impose no conditions at all. On $U_i \neq 0$ the ring is $k[y_i, u_j : j \neq i]$ with $y_j = y_i u_j$. In particular the new space is smooth, and its exceptional fibre is $\mathbb{P}^{n-1}$. For $n = 2$ the two charts are $(x, t) \mapsto (x, xt)$ and $(s, y) \mapsto (sy, y)$; on the overlap $s = 1/t$ and $y = xt$. These equations will reappear when we compute the normal bundle of the exceptional line.

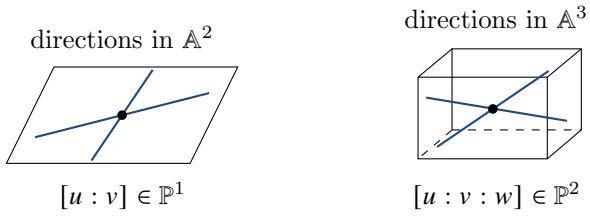


FIGURE 13. Lines through the origin in dimensions two and three, as in the lecture sketches. Each such line meets the exceptional projective space of the blow-up at exactly one point.

For example, in the chart $U_1 = 1$ the strict transform of the line $s \mapsto (s, as, bs)$ has equations $u_2 = a$, $u_3 = b$. Its intersection with the exceptional plane $y_1 = 0$ is the point $[1 : a : b]$. The directions missed by this chart have first homogeneous coordinate zero and fill a projective line. In dimension one there is just one direction, $\mathbb{P}^0$, and blowing up the origin of the affine line is the identity.

Here is the intrinsic construction which makes sense for any ideal $I = (a_1, \dots, a_n)$ of a ring A. Its *Rees algebra* is

$$\mathcal{R}_A(I) = A \oplus IT \oplus I^2T^2 \oplus \cdots \subseteq A[T], \qquad \mathrm{Bl}_I(\operatorname{Spec} A) = \operatorname{Proj} \mathcal{R}_A(I).$$

The variable T records the degree: $a \in A$ in degree zero and aT in degree one play different roles. The graded map $A[U_1, \dots, U_n] \twoheadrightarrow \mathcal{R}_A(I)$ sending U_i to $a_i T$ realizes the blow-up as a closed subscheme of $\operatorname{Spec} A \times \mathbb{P}^{n-1}$. The projection π is projective. For the origin in affine space the kernel is generated by the incidence equations above, as can also be seen on the standard charts. For a general ideal there can be additional relations. When A is a domain and $a_i \neq 0$, the ith chart has ring

$$A[I/a_i] = A[a_1/a_i, \dots, a_n/a_i] \ \subseteq \operatorname{Frac} A.$$

Taking this subring automatically cancels spurious components introduced by multiplying through by denominators. The extended ideal is (a_i) on this chart. Away from $V(I)$ one generator is a unit, so the blow-up is an isomorphism. More generally the blow-up of an invertible ideal is an isomorphism, as is seen by choosing a local generator. These properties lead to the universal property: a morphism on which the centre pulls back to an effective Cartier divisor factors uniquely through π [Har77, Ch. II, §7].

Proposition 6.6 (The exceptional scheme of a centre)**.** *For $I \subset A$, let $Z = \operatorname{Spec}(A/I)$ and $\pi : \operatorname{Bl}_I(\operatorname{Spec} A) \to \operatorname{Spec} A$. Then*

$$\pi^{-1}(Z) = \operatorname{Proj}_Z\left(\bigoplus_{j\geq 0} I^j/I^{j+1}\right).$$

If the immersion is regular of codimension $c \geq 1$ in a Noetherian scheme, this is the projective bundle of lines in the normal bundle, with fibres $\mathbb{P}^{c-1}$.

Proof. Relative Proj commutes with base change. Tensor the Rees algebra with A/I: its degree-j piece becomes I^j/II^j. Multiplication descends to multiplication of initial forms, giving exactly the graded algebra in the formula. For an ideal locally generated by a regular sequence the canonical map $\operatorname{Sym}_{A/I}(I/I^2) \to \bigoplus I^j/I^{j+1}$ is an isomorphism and I/I^2 is locally free of rank c [Sta26, Regular immersions, Tag 0638]. The relative Proj of this symmetric algebra parametrizes one-dimensional quotients of the conormal bundle, equivalently lines in its dual. For smooth centres see also [Sta26, Tag 0FUT]. □

Example 6.7 (Blowing up a whole line)**.** For the z-axis in $\mathbb{A}^3$ the ideal is $I = (x, y)$. The two charts have coordinates (x, t, z) with $y = xt$, and (s, y, z) with $x = sy$. The variable z plays no part in the substitution, so the exceptional scheme is $\mathbb{A}^1_z \times \mathbb{P}^1_{[U:V]}$. A point of it specifies both the point of the centre and a transverse direction there. Keeping z fixed recovers the familiar blow-up of a plane. For a general smooth centre these products exist locally and the normal bundle supplies their transition functions.

Proposition 6.8 (The lifting property, in coordinates)**.** *Let $g : Y \to \operatorname{Spec} A$ be a morphism for which $I\mathcal{O}_Y$ is locally generated by one nonzerodivisor. There is a unique morphism $Y \to \operatorname{Bl}_I(\operatorname{Spec} A)$ over $\operatorname{Spec} A$.*

Proof. Near a point write $g^*a_i = hb_i$, where h is a local generator of the ideal. The b_i generate the unit ideal because $(hb_i)_i = (h)$ and h is a nonzerodivisor. Choose b_j invertible locally. The required affine coordinates on the jth chart are b_i/b_j. Every homogeneous Rees relation of degree d vanishes on the b_i: multiply it by h^d, use the relation among the g^*a_i, then cancel the nonzerodivisor. The chart maps therefore exist. On an overlap they agree since both sets of ratios differ by an invertible common factor. Conversely, a lift has these ratios wherever b_j is invertible: the jth homogeneous coordinate must be invertible there, and its chart equations give the

ratios after cancellation. This proves uniqueness and shows directly why a different choice of generator h gives the same lift. □

If $I = \mathfrak{m}$ is the maximal ideal of a rational point, the exceptional fibre is Proj $(\bigoplus_{r\geq 0} \mathfrak{m}^r/\mathfrak{m}^{r+1})$, its projectivized tangent cone. It sits inside the projectivized tangent space. Thus for a singular subvariety of $\mathbb{A}^n$ one should expect only those directions that its own equations permit.

End of lecture 7

6.3. **Strict transform, total transform and the nodal cubic.** Let $Y \subset \mathbb{A}^n$ be an integral closed subvariety through the origin, of positive dimension. In the blow-up of the ambient space, the closure of the inverse image of $Y \setminus \{0\}$ is its *strict transform*. It is the blow-up of Y in the restricted ideal. On the ith chart both coordinate rings are generated in $k(Y)$ by the affine coordinates and their ratios to y_i when y_i does not vanish identically. This also explains the difference from taking the full inverse image, the *total transform*. More generally blow-ups commute with flat base change: tensoring the Rees algebra with a flat algebra identifies its graded terms with the powers of the extended ideal. Both localization and Noetherian local completion are examples.

For the cubic $y^2 = x^2(1+x)$ in characteristic different from two, use the x-chart $y = xt$. Its total transform has equation $x^2(t^2-1-x) = 0$. The strict transform is the component $x = t^2 - 1$, parametrized by

$$t \longmapsto (x, y) = (t^2 - 1, t(t^2 - 1)).$$

This is also the intrinsic chart $A[y/x]$, since $t^2 = 1+x$ there. The exceptional points are $t = 1, -1$. On the other chart put $x = sy$. Removing y^2 yields $1-s^2-s^3y = 0$, where s is a unit. This chart is Spec$k[s, s^{-1}]$ and is contained in the first chart by $s = 1/t$. Thus the entire blown-up curve is an affine line. The factor x^2 in the ambient total transform counted the multiplicity of the cubic at the centre; it does not give a double exceptional component of the intrinsic curve.

One can see the two exceptional points before choosing a chart. Write $A = k[x, y]/(y^2 - x^2(1+x))$ and send U, V to xT, yT in the Rees algebra. Besides $xV - yU$, its kernel contains

$$yV - x(1+x)U, \qquad V^2 - (1+x)U^2.$$

The last relation comes from the cubic equation in degree two. Over $x = y = 0$ the incidence equation alone would give all of $\mathbb{P}^1$. The quadratic relation restricts it to $[1 : 1]$ and $[1 : -1]$, as the chart computation confirms. In the plane these points both mapped to the origin. On the blow-up their positions retain the two different slopes there.

Proposition 6.9 (Equations of the intrinsic blow-up of the node)**.** *The three displayed Rees relations, including $xV - yU$, generate the whole kernel over A. In particular the fibre has scheme-theoretic length two.*

Proof. The kernel is homogeneous. There is no degree-zero relation, since the Rees algebra has degree-zero piece A. In degree $d \geq 1$, reduce every occurrence of V^2 using $V^2 = (1+x)U^2$. A homogeneous polynomial becomes $U^{d-1}(aU+bV)$. Its image in $A[T]$ vanishes precisely when $ax+by=0$, since A is a domain and $x \neq 0$. Write $a = a_0 + ya_1$, $b = b_0 + yb_1$ with $a_i, b_i \in k[x]$, using monic division by the cubic equation in y. Separating the coefficients of 1 and y in the syzygy gives

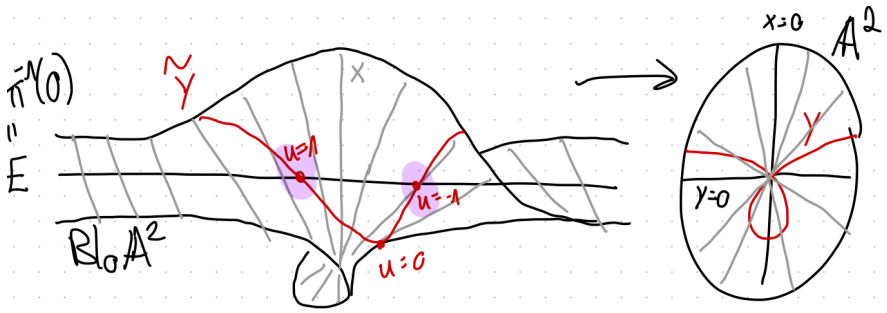
$$xa_0 + x^2(1+x)b_1 = 0, \qquad xa_1 + b_0 = 0.$$

Consequently $aU + bV = a_1(yU - xV) + b_1(yV - x(1+x)U)$. Each step was reduction by one of the proposed generators. On tensoring with $A/(x,y)$ the two linear relations vanish and one gets $\operatorname{Proj} k[U,V]/(V^2 - U^2)$. The chart $U \neq 0$ covers it: no projective point has both $U = 0$ and $V^2 = 0$. There the algebra is $k[t]/(t^2-1)$. Its two roots are simple in the characteristic under consideration. □

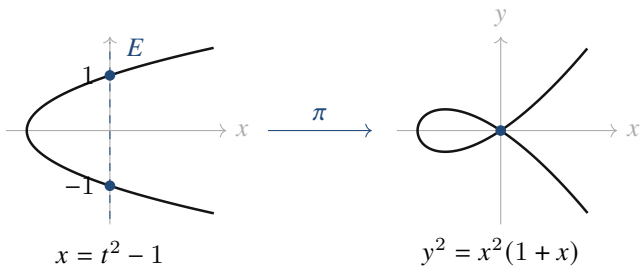
It helps to put both curves into the same drawing. In the x-chart the ambient blow-up is the surface $y = xt$ in three-space with coordinates (x, y, t). The strict transform is the space curve

$$t \longmapsto (t^2 - 1,\ t(t^2-1),\ t).$$

Projection forgets the height t. The vertical line over $(0,0)$ is the affine part of the exceptional projective line, and the two points at heights $+1$ and -1 cast the same point in the plane. The first drawing uses this projection; the next shows its source and target in their separate affine coordinates. Compare [Har77, Ch. I, Example 4.9.1].



The projection of the strict transform of the nodal cubic. The marked points lie over its two tangent directions (drawing from the lectures).



$x = t^2 - 1$ $\qquad$ $y^2 = x^2(1+x)$

For the cusp $y^2 = x^3$, the same substitution gives $x = t^2$, $y = t^3$. The strict transform is smooth after one blow-up, but meets the ambient exceptional line $x = 0$ with contact two. Thus resolving a curve intrinsically and making its union with the ambient exceptional divisor a simple normal crossings divisor are different tasks.

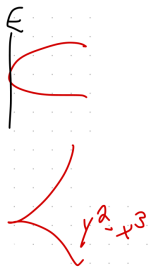


FIGURE 14. The cusp below and its first strict transform above, from the eighth lecture. The transform is smooth at its unique point on E, where its tangent is E itself. The calculation below follows this point through the next two blow-ups.

Exercise 6.10 (An embedded resolution of the cusp)**.** Blow up the plane until the reduced total transform of $y^2 = x^3$ is a divisor with simple normal crossings. Label each component by its multiplicity in the full pullback of the original equation.

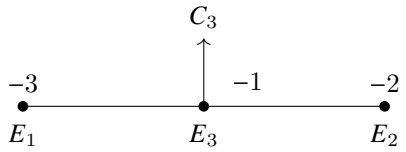
Solution. After $y = xt$ the total transform is $x^2(t^2 - x)$. The strict curve C_1 is smooth with tangent $x = 0$ at $(0, 0)$. The first exceptional curve E_1 has multiplicity two. Blow up their point of contact in the chart $x = tu$. The equation becomes

$$(tu)^2(t^2 - tu) = t^3u^2(t - u).$$

Now the old exceptional curve has strict transform $u = 0$, the new one E_2 has equation $t = 0$, and the strict cusp has equation $t = u$. There are three smooth branches with three distinct tangent lines through a *single* point. On a surface a simple normal crossings divisor has at most two branches at any point. On the other chart there are no other failures, since away from the centre the blow-up was an isomorphism and the common tangent direction was unique.

Blow up this triple point. On $u = tv$ one obtains $t^6v^2(1 - v)$, while on $t = uw$ one obtains $u^6w^3(w - 1)$. The new exceptional E_3 has multiplicity six. Its three marked points have directions $[t : u] = [1 : 0], [0 : 1], [1 : 1]$. They are the intersections with E_1, E_2, C_3, respectively. Each is transverse by the chart equations. The self-intersection of a smooth curve falls by one when one blows up a point of the curve [Har77, Ch. V, Proposition 3.2]. Starting each new exceptional at -1 therefore gives the diagram below. The

strict transforms of the cusp at the three successive centres had multiplicities $2, 1, 1$. Only the first contributed to δ in Proposition 5.8.



□

6.4. **The first blow-up of a double point.** For A_n use the equation $xy+z^{n+1}=0$. In the x-chart the strict transform has equation $v+x^{n-1}w^{n+1}=0$ with $y=xv, z=xw$; the y-chart works symmetrically. In the z-chart put $x=zu, y=zv$. The only equation is $uv+z^{n-1}=0$. For $n=1$ or 2 this chart is smooth. For $n\geq 3$ the residual singularity is A_{n-2}. When $n=1$ the exceptional fibre is a smooth projective conic; for $n\geq 2$ it is two lines meeting in the direction of the z-axis. This explains the change by two in the subscript.

One can do all five families by the same chart calculation. Here we use the classical normal forms over $\mathbb{C}$. Only singularities on the strict transform are listed.

Original point	Singularities after the first blow-up
A_1, A_2	none
$A_n,\ n\geq 3$	A_{n-2}
D_4	three A_1
D_5	A_3 and A_1
$D_n,\ n\geq 6$	D_{n-2} and A_1
$E_6,\ E_7,\ E_8$	$A_5,\ D_6,\ E_7$, respectively

To check the table for D_n substitute $x=ys, z=yv$ into $z^2+x^2y+y^{n-1}$ and divide by y^2. The equation is $v^2+ys^2+y^{n-3}$. For $n=4$ it gives nodes at the two simple roots of s^2+1. For $n=5$, completing the square in y gives two squares and a fourth power, hence A_3. The direction $[1:0:0]$, visible on the x-chart alone, gives an additional node, since its quadratic part is v^2+xt. For the E cases the x-chart is smooth along the exceptional curve away from the y-chart, and the z-chart has no exceptional point. The y-chart gives respectively

$$v^2+ys^3+y^2,\quad v^2+ys^3+y^2s,\quad v^2+ys^3+y^3.$$

Complete the square in y in the first two equations. The residual terms are a sixth power in the first and a fifth power in the second with s multiplying the square. The last equation is already E_7 after renaming variables.

Example 6.11 (Where the three arms of D_4 start)**.** In the y-chart of the first blow-up the equation is $v^2+y(s^2+1)=0$. Along $y=v=0$ it is smooth

if s^2+1 is invertible, since one can solve for y. At either of the roots $s_0 = \pm i$ write $S = s - s_0$; then $s^2 + 1 = S(2s_0 + S)$. The second factor is a unit. Replacing y by $y(2s_0 + S)$ gives $v^2 + yS = 0$, an A_1 point. In the other chart $y = xt$, $z = xw$, the strict equation is $w^2+xt(1+t^2) = 0$. Directions with $t \neq 0$ are already in the y-chart; $t = 0$ gives a third A_1, since $1+t^2$ is a unit there. The double line defined by the initial form z^2 has the underlying space $\mathbb{P}^1$. Blowing up the three distinct nodes gives one smooth conic over each of its marked points. These are the three ends attached to the central curve in the D_4 resolution graph.

For comparison take E_6. The residual equation $v^2 + ys^3 + y^2 = 0$ on the y-chart is

$$v^2 + (y + s^3/2)^2 - s^6/4 = 0.$$

Set $Y = y+s^3/2$. The two coordinates $v+iY$ and $v-iY$ have independent linear parts. Their product is $s^6/4$, the equation of A_5. In type E_7 the equation is $v^2 + s(y^2 + ys^2)$; completing the square yields $v^2 + s(y + s^2/2)^2 - s^5/4$. This is the D_6 equation. Both changes require two to be a unit; using the same normal forms in bad characteristic would lose some analytic types.

6.5. **Recognizing a singular chart.**

Theorem 6.12 (Jacobian criterion at a rational point)**.** *Let a be a k-rational point of the affine scheme with coordinate ring*

$$A = k[X_1, \dots, X_n]/(f_1, \dots, f_r).$$

The dimension of its tangent space is $n - \operatorname{rank} J(a)$. It is regular exactly when this number equals $\dim A_{\mathfrak{m}_a}$.

Proof. Write $\mathfrak{m}$ for the maximal ideal of a. The classes of $X_i - a_i$ span $\mathfrak{m}/\mathfrak{m}^2$. The linear relations among them are the linear parts of all elements of the defining ideal. Since $f_i(a) = 0$, the linear part of $\sum h_i f_i$ is $\sum h_i(a)\, df_i(a)$. Thus

$$\operatorname{edim} A_{\mathfrak{m}} = n - \operatorname{rank}\left(\frac{\partial f_i}{\partial X_j}(a)\right).$$

Regularity means equality with the local dimension. In particular at a closed point of a hypersurface over an algebraically closed field it is equivalent to the nonvanishing of at least one partial derivative. □

For the nodal cubic, $f_x = -3x^2-2x$ and $f_y = 2y$. Imposing $f = f_x = f_y = 0$ leaves only the origin in characteristic different from two.

Corollary 6.13. *Over an algebraically closed field the Jacobian test at every closed point tests regularity everywhere.*

Proof. Each prime is contained in a maximal ideal. Localizing a regular local ring gives a regular local ring [Sta26, Tag 00NN], and the weak Nullstellensatz makes all closed points rational over this field. □

Over an imperfect field the Jacobian detects smoothness over the base; a regular point can have inseparable residue extension. For example $\operatorname{Spec} k[T]/(T^p - a)$ for $a \notin k^p$ is the spectrum of a field and is not smooth over k [Sta26, Tag 07EK].

Exercise 6.14 (A node whose two directions require a field extension)**.** Fix a field k of characteristic different from two and let C_a be the curve $y^2 = x^2(x + a)$, $a \in k$. Find the singular points, identify the completed equation at the origin, and compute the exceptional fibre of the first blow-up.

Solution. After any field extension the equations $f = f_y = 0$ imply $y = 0$ and either $x = 0$ or $x = -a$. For $a \neq 0$ the derivative $f_x(-a, 0) = -a^2$ excludes the second point. Thus the origin is the only geometric singular point in every case. If $a = 0$ it is a cusp. If $a \neq 0$, Hensel gives a unit $h(x)$ with constant term one and $h(x)^2 = 1 + x/a$. With $Y = y/h(x)$ the equation becomes $Y^2 = ax^2$. This is a split node precisely when a is a square in k.

On the blow-up chart $y = xt$, the strict transform is $t^2 = x + a$. It is an affine line with coordinate t. The omitted direction $[0 : 1]$ does not lie in the tangent cone, so the whole exceptional fibre has algebra $k[t]/(t^2 - a)$. It consists of two rational points for a a nonzero square, a single closed point with separable quadratic residue field for a a nonsquare, and a doubled rational point for $a = 0$. In particular, after base extension the geometric branches can separate even though no second rational point appeared over the original field. □

Example 6.15 (An isolated point becoming a whole singular line)**.** Take $k = \mathbb{C}$ and $S = V(z^2 + x^3y + y^5)$. Its partial derivatives are $2z, 3x^2y, x^3 + 5y^4$, whose common zero is the origin. The three charts of its strict transform at that point have equations

$$\begin{aligned} x\text{-chart}: &\quad w^2 + x^2v + x^3v^5 = 0, \\ y\text{-chart}: &\quad w^2 + u^3y^2 + y^3 = 0, \\ z\text{-chart}: &\quad 1 + u^3vz^2 + v^5z^3 = 0. \end{aligned}$$

For instance $x = yu, z = yw$ gives the middle equation after division by y^2. All three partials of the middle equation vanish on $y = w = 0$, with u arbitrary. In the first chart the analogous line is $x = w = 0$; the third chart misses the exceptional fibre. The reduced exceptional fibre is a whole projective line of singular points. The exceptional fibre as a scheme is the double line cut out by the initial form z^2. Thus point blow-ups need not preserve normality, even when the original surface has an isolated hypersurface singularity.



FIGURE 15. Two real plots from the lecture. Left: $z^2 + x^3y + y^5 = 0$. Right: the y-chart after blowing up the origin, in which the singular locus is the line $y = w = 0$. The calculation detects the full line independently of what a real viewpoint reveals.

To continue resolving this surface we need an operation that repairs codimension-one singularities. Its algebraic definition uses integrality in the total ring of fractions.

End of lecture 8

7. Normalization and resolution

7.1. **Putting missing integral functions back.** Let A be reduced and Noetherian. Its total ring of fractions $Q(A)$ is obtained by inverting the nonzerodivisors. If $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the minimal primes, it is the product $\prod_i \operatorname{Frac}(A/\mathfrak{p}_i)$. An element b of an A-algebra is *integral over* A when it satisfies a monic polynomial with coefficients in A. Equivalently $A[b]$ is finite as an A-module: the powers up to degree one less than a monic relation span it, and conversely the determinant trick gives a monic relation for multiplication by b. Applying the same argument to $A[b, c]$ shows that sums and products of integral elements are integral.

The *normalization* $\bar{A}$ is the integral closure of A in $Q(A)$. For a domain, being normal means being integrally closed in its fraction field. For a reduced Noetherian ring it means that the local rings are integrally closed domains, equivalently $A = \bar{A}$. The idempotents of the product of component fields satisfy $e^2 - e = 0$ and belong to $\bar{A}$, so normalization separates irreducible components. Localization commutes with normalization: clear the denominators in a monic equation over $S^{-1}A$ to make a multiple of the integral element integral over A. For algebras of finite type over a field the normalization is finite. These rings are excellent; for their reduced local rings normalization also commutes with completion, with the completed normalization understood as the product over points above the given point [Sta26, Tags 035E, 0C23]. No finiteness assertion for arbitrary Noetherian rings is needed here.

Exercise 7.1 (What does passage to total fractions preserve?)**.** Suppose $A \neq 0$ is Noetherian and every zero divisor is nilpotent. Must its total ring of fractions be a field? Compare $Q(A)$ with the fraction field of A_{red}.

Solution. Write $N = \sqrt{(0)}$. If $ab \in N$ and $a \notin N$, then a is a nonzerodivisor. From $a^m b^m = 0$ one obtains $b^m = 0$, hence $b \in N$. Thus N is prime. Every element of N is a zero divisor in this nonzero ring, while its complement consists exactly of the nonzerodivisors. Consequently $Q(A) = A_N$ is a zero-dimensional Noetherian local ring, hence Artinian, and

$$Q(A)_{\mathrm{red}} = A_N/NA_N = \mathrm{Frac}(A/N).$$

The localization map $A \to Q(A)$ is injective, so a nonzero nilpotent survives. For $A = k[\epsilon]/(\epsilon^2)$ all nonzerodivisors are already units and $Q(A) = A$, although $\mathrm{Frac}(A_{\mathrm{red}}) = k$. This is where reducedness enters the component-field description used in normalization. □

Lemma 7.2 (Testing a proposed normalization)**.** *Let $A \subset B \subset Q(A)$ for a reduced Noetherian ring A. Suppose B is integral over A, has the same total ring of fractions, and is normal. Then $B = \bar{A}$.*

Proof. The first inclusion $B \subseteq \bar{A}$ is the definition of integrality. Conversely every $z \in Q(A)$ integral over A satisfies the same monic equation over B, and $Q(A) = Q(B)$. Normality of B puts z in B. In computations we will therefore look for a fraction regular away from the bad locus, write its monic equation, then check the new ring is normal. □

Exercise 7.3. Why may one invert x when looking for the normalization of the irreducible nodal cubic, but not when taking the *total* ring of fractions of $k[x, y]/(xy)$?

Solution. For the cubic $1 + x$ has a simple zero at -1, so is not a square in $k(x)$. Over a field of characteristic different from two the equation $y^2 - x^2(1 + x)$ is irreducible over $k(x)$, hence in $k[x, y]$. In its coordinate ring x is nonzero in a domain. On the reducible curve $xy = 0$, x annihilates the nonzero function y. Inverting x erases the whole y-axis. To retain the two generic points, only invert functions that are nonzero on *both* components. For example $x+y$ is such a denominator. The fractions $x/(x+y)$ and $y/(x+y)$ give the two idempotents selecting the components in $k(x) \times k(y)$. For nonreduced rings even a single component can retain additional information in its total ring of fractions: in $k[\epsilon]/(\epsilon^2)$ the nonzerodivisors are already units. This is why we started the definition with a reduced ring. □

Example 7.4 (Two curves)**.** The cusp ring is $A = k[t^2, t^3] \subset k(t)$. It lacks $t = y/x$, which satisfies $T^2 - x = 0$. Since $A[t] = k[t]$ is normal and has the same fraction field, it is the whole normalization. Indeed an element of that field integral over A is integral over $k[t]$ as well. There is one point over the cusp, and $k[t]/A$ as a vector space has basis the class of t. For the nodal cubic, write $t = y/x$, so $t^2 = x + 1$, $y = t(t^2 - 1)$. Again $A[t] = k[t]$ is the normalization, but now $t = \pm 1$ are distinct points over the node if $\mathrm{char}\, k \neq 2$. This agrees with the blow-up calculation. Locally at the node the completed normalization consists of two power series rings; a function

on the completed node is a pair of series with equal constant terms. Both curves have $\delta = 1$ and have different numbers of branches.

Example 7.5 (Eliminating the new variable, and the conductor)**.** For the node let $B = A[Z]/(xZ - y, Z^2 - 1 - x)$. There is no hidden third relation:

$$B \simeq k[Z], \qquad x = Z^2 - 1, \quad y = Z(Z^2 - 1).$$

Indeed after these substitutions $y^2 - x^2(1+x)$ becomes $(Z^2-1)^2[Z^2-1-(Z^2-1)] = 0$. The map into the curve's function field $\mathrm{Frac}(A)$ sending Z to y/x is injective because Z is transcendental over k (otherwise $x = Z^2 - 1$ would be algebraic). This verifies the kernel statement directly. A polynomial $P(Z)$ belongs to A exactly when $P(1) = P(-1)$. To see the converse, subtract this common constant and divide by $Z^2 - 1$; the quotient has the form $a(Z^2 - 1) + Zb(Z^2 - 1)$. Thus $P = c + xa(x) + yb(x)$. The conductor $\{h \in B : hB \subset A\}$ is precisely $(Z^2 - 1)B$. Vanishing at both points suffices; conversely the conditions for h and Zh to belong to A force both values to vanish. For the cusp the same definition gives $t^2 k[t]$. An ordinary node and a cusp each have a length-one normalization quotient, but their conductor cuts out two distinct points or a doubled point on the smooth curve, respectively.

A UFD is normal: if a/b in lowest terms satisfies a monic equation, clearing denominators gives $b \mid a^n$, forcing b to be a unit. In dimension one there is a stronger statement. A one-dimensional Noetherian local domain is normal if and only if it is a discrete valuation ring, equivalently if and only if it is regular [AM69, Ch. 9]. Thus normalization resolves a reduced algebraic curve over a perfect field. The same argument on a surface removes singularities in codimension one, leaving isolated points.

Theorem 7.6 (The one-dimensional case)**.** *For a one-dimensional Noetherian local domain $(R, \mathfrak{m})$, the following are equivalent: R is integrally closed, $\mathfrak{m}$ is principal, and R is a discrete valuation ring.*

Proof. Assume R is integrally closed and choose $0 \neq a \in \mathfrak{m}$. The only prime containing a is $\mathfrak{m}$, by dimension one. Since $\mathfrak{m}$ is finitely generated, some power $\mathfrak{m}^n$ is contained in (a). Choose the smallest such $n \geq 1$ and $b \in \mathfrak{m}^{n-1} \setminus (a)$. Then $z = b/a$ lies outside R, but $z\mathfrak{m} \subset R$. If $z\mathfrak{m} \subseteq \mathfrak{m}$, multiplication by z on the finite faithful module $\mathfrak{m}$ would give a monic equation for z by the determinant trick. Integral closedness would force $z \in R$, a contradiction. Hence the ideal $z\mathfrak{m}$ contains a unit and equals R, so $\mathfrak{m} = z^{-1}R$ is principal.

Write $\mathfrak{m} = (t)$. For each nonzero $h \in R$ choose the largest j with $h \in (t^j)$, using Krull intersection. Then $h = t^j u$ with u a unit. Thus every fraction has a well-defined integral order, and its order is nonnegative exactly when it belongs to R. This is a DVR. Finally, let h have negative order and suppose $h^s + c_1 h^{s-1} + \cdots + c_s = 0$ with $c_i \in R$. The leading power h^s has strictly smaller order than every other nonzero summand, a contradiction.

Hence every element of the fraction field integral over the DVR belongs to it, closing the cycle of implications. □

Corollary 7.7. *For a reduced hypersurface of finite type over a perfect field, normality is equivalent to regularity at all codimension-one points.*

Proof. A hypersurface cut out by a nonzerodivisor in a regular local ring is Cohen–Macaulay: both depth and dimension decrease by one, also after localization. In particular (S_2) holds. A reduced ring is already regular at its minimal primes. Serre's criterion therefore leaves precisely the height-one condition. In dimension two this condition says the singular locus has dimension at most zero. We apply the criterion to the individual charts below. □

7.2. **Normal surfaces can still be singular.** Over an algebraically closed field of characteristic different from two put

$$A = k[x, y, z]/(z^2 - x^2(x^2 + y^2)).$$

The polynomial $x^2 + y^2$ is square-free and is not a square in $k(x, y)$, so A is a domain. The fraction $w = z/x$ satisfies $w^2 = x^2 + y^2$. Adjoining it gives $B = k[x, y, w]/(w^2 - x^2 - y^2)$. This is finite and birational over A; its only singular point is the origin. The quadratic form is nondegenerate, so this is our A_1 cone. To see that the normalization really stops here we can use either of the following two arguments.

Proposition 7.8 (A double cover of affine space)**.** *Let $R = k[x_1, \dots, x_n]$, with $\operatorname{char} k \neq 2$, and let $f \in R$ be square-free, nonconstant. Then $R[z]/(z^2 - f)$ is a normal domain.*

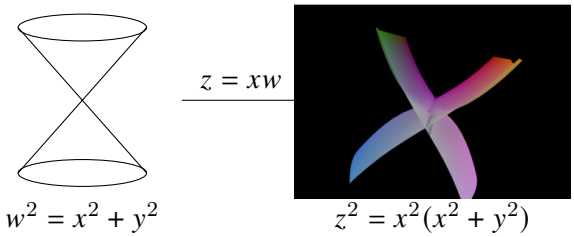
Proof. The fraction field is $K(\sqrt{f})$, $K = \operatorname{Frac} R$. If $g + h\sqrt{f}$, with $g, h \in K$, is integral over $R[\sqrt{f}]$, it is integral over R by transitivity. Its conjugate is also integral, so the trace $2g$ and norm $g^2 - h^2 f$ are in R by normality of R. Thus $g \in R$ and $h^2 f \in R$. At an irreducible denominator of h the order of h^2 would be at most -2, whereas f has order at most one. It follows that $h \in R$. This is the double-cover calculation of [Har77, Ch. II, Ex. 6.4]. □

Theorem 7.9 (Serre's criterion)**.** *A Noetherian ring is normal precisely when it satisfies (R_1) and (S_2). The first requires regularity in codimensions zero and one. The second requires $\operatorname{depth} A_{\mathfrak{p}} \geq \min(2, \dim A_{\mathfrak{p}})$ for every prime $\mathfrak{p}$* [Sta26, Tag 031O].

Proof. This is Serre's normality criterion [Sta26, Tag 031S]. In the forward direction the localizations at height one are DVRs and normal Noetherian domains satisfy (S_2). For the reverse implication (R_1) gives normality in codimension one, while (S_2) yields extension across codimension at least two. The cited proof treats reduced rings with several components as well. □

Depth counts the length of a longest regular sequence in the maximal ideal. A hypersurface in a regular scheme is Cohen–Macaulay (its depth at each point equals its dimension), as is any local complete intersection. For a reduced hypersurface it is therefore enough to check the height-one points. This proves normality both for B and for the original surface of Example 6.15. As another comparison the pinch point $z^2 = x^2y$ has normalization $k[x, w]$ with $y = w^2, z = xw$; here adjoining the missing function does make the surface smooth.

Example 7.10 (The fibres along the singular line)**.** For $A = k[x, y, z]/(z^2 - x^2(x^2 + y^2))$ the normalization map is $(x, y, w) \mapsto (x, y, xw)$. Outside $x = 0$ its inverse is literally $w = z/x$. Over a point $(0, b, 0)$ the algebra of the fibre is $k[w]/(w^2 - b^2)$. If $b \neq 0$ the roots $b, -b$ correspond to two separate sheets. At $b = 0$ they coalesce into a length-two fibre. In particular finiteness and birationality do not require that every closed fibre consist of one reduced point. Over the real numbers the normalization $w^2 = x^2 + y^2$ is the familiar double cone. The map to the original surface multiplies its height w by x and identifies opposite heights along $x = 0$. This describes the real sketches while the two algebras describe the construction over the field.



The normalization map; a real view of the target from the lecture notes.

We can now return to the singular line created in Example 6.15. On its y-chart the equation was $w^2 + y^2(u^3 + y) = 0$. The missing fraction $s = w/y$ is integral, with $s^2 + u^3 + y = 0$. Adjoining it gives $k[u, s]$, since $y = -s^2 - u^3$. This polynomial ring is normal and has the same fraction field, so it is the normalization of that entire chart. Over the exceptional line the new curve has equation $s^2 + u^3 = 0$. The ambient chart has become smooth, although the reduced inverse image of the line has a cusp. This is why a resolution and an embedded normal crossings resolution keep track of different things.

Exercise 7.11 (Normalizing the Whitney umbrella)**.** Over an algebraically closed field of characteristic different from two, let X have equation $z^2 = x^2y$. Find its singular locus and normalization. What happens to the two sheets over the singular line when they reach the origin?

Solution. The derivatives are $-2xy, -x^2, 2z$. Their simultaneous vanishing on X gives $x = z = 0$, with y free. The coordinate ring A is a domain, since y is not a square in $k(x, y)$. In its fraction field set $w = z/x$. Then

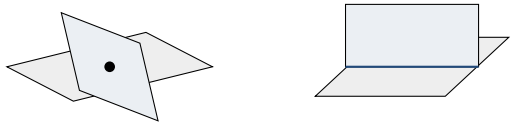
$w^2 = y$ and $z = xw$, hence $A[w] = k[x, w]$. The latter is polynomial: x, w are algebraically independent since its fraction field contains the algebraically independent x, y. It is integral over A, is birational to A and is normal, so it is the normalization. The map is $(x, w) \mapsto (x, w^2, xw)$.

Over $(0, b, 0)$ its fibre has ring $k[w]/(w^2 - b)$. For $b \neq 0$ this is a product of two copies of k. For $b = 0$ it has one point of length two. There is a distinction from the preceding example $z^2 = x^2(x^2+y^2)$: the inverse image of the singular line in the normalization here is one affine line with parameter w, covering the y-line by $w \mapsto w^2$. In the preceding normalization the preimage has equation $w^2 = y^2$ and already has two irreducible components. The fibre over the pinch does not by itself tell these two situations apart. □

Example 7.12 (Two planes joined at a point)**.** Consider the affine algebra

$$A = k[x, y, z, w]/(xz, xw, yz, yw).$$

Its map to $B = k[x, y] \times k[z, w]$ identifies A with the pairs of polynomials having equal values at the two origins. Off the origin each plane is regular. Nevertheless $e = (1, 0)$ is missing; it satisfies $e^2 - e = 0$ and $A[e] = B$. Thus B is the normalization. The obstruction to normality was concentrated in codimension two. Here the depth at the vertex can actually be computed. In the local ring, $\ell = x + z$ corresponds to (x, z) and is a nonzerodivisor. In the quotient by ℓ the class of $x = (x, 0)$ is nonzero: an equation $(x, 0) = \ell(a, b)$ would require $a = 1$ and $b = 0$, whose values at the origins do not agree. But the maximal ideal annihilates this class. For (a, b) vanishing at the origins one has $(a, b)(x, 0) = \ell(a, 0)$ with $(a, 0)$ in the ring. Thus there is no second nonzerodivisor after ℓ, so the depth is one. Geometrically a regular function on the punctured neighbourhood may take the constant value one on one plane and zero on the other. The failure to extend it across the origin is the geometric obstruction measured here by (S_2).



one common point a common line

FIGURE 16. Two different gluings of affine planes. In the left-hand schematic view the planes are in four-space and share only their marked origins. On the right the common line is a codimension-one locus.

Exercise 7.13 (Two planes meeting along a line)**.** Compare the example above with $A' = k[x, y, z]/(xy)$. Test both of Serre's conditions.

Solution. The planes now meet along $x = y = 0$, which has codimension one in their union. At its generic point the maximal ideal of the localization has generators x, y whose images are independent modulo squares over $k(z)$.

Its dimension is one, so (R_1) fails. At the origin z is a nonzerodivisor; after quotienting by z, $x + y$ is a nonzerodivisor in $k[x, y]_{(x,y)}/(xy)$, as is seen on the two components. This gives depth two. More generally this hypersurface is Cohen–Macaulay at every point, so (S_2) holds everywhere. Its normalization is $k[x, z] \times k[y, z]$ by Lemma 7.2, with the old subring consisting of pairs agreeing on the entire z-axis. In the example of the two planes meeting only at a point, the failure was exactly the other way around. Thus a condition just on the codimension of the singular locus cannot test normality for arbitrary reduced schemes. □

7.3. **The surface resolution theorem.** A *resolution* of an integral surface Y is a proper birational morphism $\pi : X \to Y$ with X regular; we choose it to be an isomorphism on the regular locus. Over a perfect field regular varieties are smooth. A morphism to an affine base is projective if it factors through a closed immersion into projective space over that base followed by projection. Both the Rees construction and finite normalization are projective. We can therefore ask for a projective resolution in the following result.

Theorem 7.14 (Resolution of excellent surfaces)**.** *A reduced excellent surface has a resolution by a finite sequence of normalizations and blow-ups at closed points, which is an isomorphism over the regular locus.*

Proof. For termination we invoke Lipman's desingularization theorem [Lip78]; for the precise successive normalized blow-up construction see [Sta26, Resolution of Surfaces, Theorem 14.5]. The centre at each stage on a normal surface is zero-dimensional: the regular locus contains all height-one points by (R_1) and is open by excellence. The singular locus is closed, Noetherian and zero-dimensional, thus finite on our quasi-compact surfaces. Normalization is finite under the same hypotheses. Each step is therefore proper and birational and changes nothing on the regular open. Termination is the substantive additional assertion of the theorem. The local calculation following Example 6.15 explains why the normalization in its statement is part of the construction. □

The problem already looks different if one embeds the surface in a smooth threefold. Keeping the ambient space smooth at each step is possible by blowing up smooth centres; requiring the final exceptional and strict transforms to cross normally is then an additional geometric condition. The embedded cusp calculation was a small model of that distinction. One can even read the old equation on each new chart and track separately the multiplicities of the exceptional components. The higher-dimensional theorem explains what one can require of the ambient space and the transformed subvariety simultaneously.

Theorem 7.15 (Embedded resolution in characteristic zero)**.** *Let Y be an integral closed subvariety of a smooth variety M over a field of characteristic*

zero. There is a sequence of blow-ups of the smooth ambient variety along smooth centres such that the final strict transform of Y is smooth and has normal crossings with the exceptional divisor. The modification may be chosen to be an isomorphism over the smooth locus of Y in its restriction to Y.

Proof. This is the embedded form of Hironaka's resolution theorem [Hir64, Parts I and II]; for a formulation with centres selected by local invariants see [FH10]. To read the last transversality assertion when Y is a hypersurface, choose local parameters in the final ambient variety. The reduced support of its total transform is given by the product of a subset of those parameters. Multiplicities in the total transform can still exceed one, exactly as in the cusp calculation. □

A historical interlude. For a surface in three-space, a projection to a plane presents the surface locally as a branched cover. One can resolve the plane branch curve by point blow-ups, but then one still has to understand the induced covering singularities and glue the local modifications. This distinguishes two parts of the old surface problem that can look identical in a drawing. Segre and Beppo Levi discussed the decomposition of multiple points in the 1890s; later accounts by Severi and Albanese pursued birational transformations to nonsingular models [Seg97, Lev97, Sev14, Alb24]. For the distinction between the Italian arguments, Jung's local method of 1908 and Walker's global proof in the 1930s one may consult Lipman's historical introduction [Lip75, Lecture 1]. The valuation-theoretic approach asks for a model on which a specified valuation has a regular centre. It is a local uniformization problem with local now referring to the valuation. Zariski's surface paper appeared in 1939 [Zar39]. We will need only the geometric theorem and the elementary discrete valuations attached to curves on a resolved surface.

For the original surface proofs of Walker and Zariski see [Wal35, Zar39]. Abhyankar treated surfaces in positive characteristic [Abh56]. Hironaka's theorem in characteristic zero allows smooth blow-up centres in arbitrary dimension [Hir64]. In positive characteristic Cossart and Piltant proved resolution for threefolds over, in particular, perfect fields [CP09]. The surface algorithm above is a theorem in dimension two; it is not a general recipe for resolving higher-dimensional varieties. For the rest of the course we will ask what the curves on a resolution tell us about the point below them.

End of lecture 9

7.4. **Blowing up the vertex of a cone.** Which normal surfaces can be resolved by point blow-ups alone? We have already found the A_n. It is tempting to guess that this property should force the exceptional curves to be rational. The following construction lets us prescribe their genus.

Let $C = V_+(f) \subset \mathbb{P}^2_k$ be a smooth integral plane curve of degree d, with k algebraically closed, and put $S = \operatorname{Spec} k[x,y,z]/(f)$. The same homogeneous

polynomial describes two different spaces: a projective curve of directions and a surface swept out by the corresponding lines. In $\mathbb{P}^3$ the vertex is $[0:0:0:1]$, and the line to $[a:b:c:0]$ consists in the affine chart of the points $\lambda(a,b,c)$. Outside the vertex the surface is smooth; for example on $x \neq 0$ it is the product of a chart of C and $\mathbb{G}_m$. Being a hypersurface regular in codimension one, S is normal.

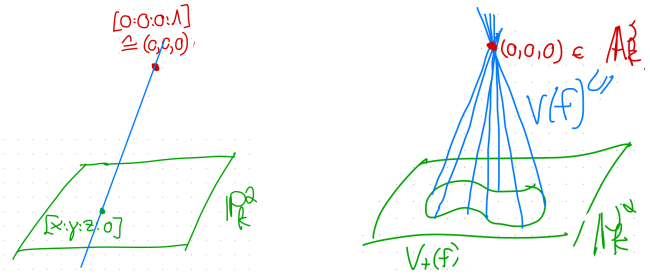


FIGURE 17. From one direction to the entire cone, in two drawings from the tenth lecture. On the left the line through $[0:0:0:1]$ meets the plane at infinity in one point. On the right these points range over $V_+(f)$. After blowing up, that projective curve records precisely which line one followed to the vertex.

Proposition 7.16 (Resolving the plane cone)**.** *For a smooth integral plane curve C the blow-up of its cone at the vertex is smooth. Its exceptional curve maps isomorphically to C by the direction map.*

Proof. Blow up the vertex. In the x-chart put $y = xv, z = xw$. Homogeneity gives

$$f(x, xv, xw) = x^d f(1, v, w).$$

The symbol x in this chart now has two roles: its vanishing says that the original point was the vertex, whereas $[1:v:w]$ remains a coordinate on the space of directions even at $x = 0$.

Removing the exceptional factor gives the ring $k[x,v,w]/(f(1,v,w))$. The exceptional curve on that chart is simply $x = 0$. The other two charts have the same description, so the exceptional curve is C itself, with its reduced scheme structure even when $d > 1$. Notice the distinction: d was the multiplicity in the *ambient total transform*. The strict transform is smooth since each chart is a product of a smooth affine curve with an affine line. □

Proposition 7.17 (The exceptional scheme of a homogeneous cone)**.** *Let $A = k[X_0, \dots, X_n]/J$ for a homogeneous ideal J, put $\mathfrak{n} = A_+$, and let $C = \operatorname{Proj} A$. The exceptional fibre of the blow-up of $\operatorname{Spec} A$ at its vertex is canonically C, including its scheme structure.*

Proof. The k-algebra A is generated in degree one, whence $\mathfrak{n}^j = \bigoplus_{r \geq j} A_r$. Taking degree j therefore gives a vector-space isomorphism $\mathfrak{n}^j/\mathfrak{n}^{j+1} \simeq A_j$.

These maps commute with multiplication. Base change of the relative Proj of the Rees algebra to $A/\mathfrak{n}$ gives

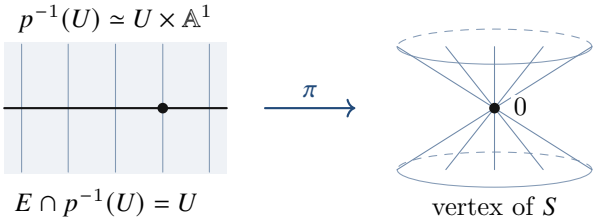
$$\operatorname{Proj}\left(\bigoplus_{j\geq 0}\mathfrak{n}^j/\mathfrak{n}^{j+1}\right)\simeq \operatorname{Proj} A.$$

In particular a smooth plane curve gives a smooth exceptional curve. A doubled line would instead give a doubled line as the exceptional fibre. Homogeneity is essential for identifying this fibre with the projective curve defined by the original equations; in general one must take their initial ideal. □

Here is the gluing behind the plane-curve calculation. Write x_0, x_1, x_2 for coordinates on the cone and $[U_0 : U_1 : U_2]$ for direction coordinates. On $U_i \neq 0$ put $a_j = U_j/U_i$ for $j \neq i$, and set $a_i = 1$. The chart is

$$k[x_i, a_j \ (j \neq i)]/(f(a_0, a_1, a_2)), \qquad x_j = x_i a_j.$$

On $U_j \neq 0$ the coordinates transform by $x_j = a_j x_i$ and $a'_\ell = a_\ell/a_j$. These equations remain meaningful at $x_i = 0$ as long as $a_j \neq 0$. They glue the exceptional curves by precisely the ordinary transition functions of $V_+(f)$. They also explain the incidence description in the notebooks: in $\mathbb{P}^3$ the line joining $[0:0:0:1]$ to $[a:b:c:0]$ consists of $[\lambda a : \lambda b : \lambda c : \mu]$. In the affine chart $\mu = 1$ this runs through $(\lambda a, \lambda b, \lambda c)$, including the vertex. The blow-up remembers the chosen point $[a : b : c]$ when λ tends to zero.



Each vertical fibre on the left maps to one generating line on the cone. Their zero vectors have different labels in C and have the same image in S.

There is a global description that remembers how the products glue. A point over $[a : b : c] \in C$ is a vector in the line of k^3 spanned by (a, b, c). Thus the blown-up surface is the total space of the tautological line bundle $\mathcal{O}_C(-1)$, and its exceptional divisor E is the zero section. For instance, when we change from the x-chart to the y-chart the fibre coordinate changes by $y = vx$. The normal bundle of the zero section is the line bundle itself, hence

$$E^2 = \deg \mathcal{O}_C(-1) = -d.$$

For a conic this gives $\mathbb{P}^1$ with self-intersection -2, just what we found for A_1. For a smooth cubic it gives a genus-one curve of self-intersection -3. More generally $g(C) = (d-1)(d-2)/2$ by plane adjunction. The single blow-up resolves the cone in all these cases; whether the resolved singularity is *rational* is quite another matter.

We can already see the obstruction in cohomology. The projection $p : X = \operatorname{Tot} \mathcal{O}_C(-1) \to C$ is affine, and $p_*\mathcal{O}_X = \bigoplus_{j \geq 0} \mathcal{O}_C(j)$. Hence $H^1(C, \mathcal{O}_C)$ is a direct summand of $H^1(X, \mathcal{O}_X)$. Over the affine cone this contributes to $R^1\pi_*\mathcal{O}_X$ at the vertex. It survives for a smooth cubic. For a conic every $\mathcal{O}_C(j)$ has nonnegative degree on $\mathbb{P}^1$, so $H^1(X, \mathcal{O}_X) = 0$. This is the cohomological difference that the word rational will capture.

Example 7.18 (The conic, including characteristic two)**.** Let C be $XZ = Y^2$. The map $[s : t] \mapsto [s^2 : st : t^2]$ identifies it with $\mathbb{P}^1$ over every field: Y/X and Y/Z are inverse affine coordinates on the overlap, and $X = Z = 0$ gives no point on C. The cone chart over $X \neq 0$ in the direction space is $\operatorname{Spec} k[a, u]$, with $(x, y, z) = (a, au, au^2)$. Over $Z \neq 0$ it is $\operatorname{Spec} k[b, v]$, with $(x, y, z) = (bv^2, bv, b)$. The gluing reads $v = u^{-1}$ and $b = u^2 a$. This is the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$: a point over the conic is a vector in the line spanned by $(1, u, u^2)$. In particular the conic is smooth even in characteristic two, since the partials with respect to X, Z cannot both vanish at a point of C. No extraction of a square root was used.

Proposition 7.19 (Geometric genus of the plane cone)**.** *Let C be a smooth plane curve of degree d over an algebraically closed field. For the resolution $\pi : X \to Y$ of its affine cone above, one has*

$$\dim_k H^1(X, \mathcal{O}_X) = \binom{d}{3}.$$

In particular the vertex is rational for $d = 2$ and fails to be rational for every $d \geq 3$.

Proof. The affine bundle projection gives $\bigoplus_{j \geq 0} H^1(C, \mathcal{O}_C(j))$, since cohomology on this separated curve can be computed on a finite affine cover whose intersections are affine. By adjunction $\omega_C = \mathcal{O}_C(d - 3)$; by Serre duality the jth summand is dual to $H^0(C, \mathcal{O}_C(d - 3 - j))$. It vanishes for $j > d - 3$. In the remaining range set $a = d - 3 - j$. A section of $\mathcal{O}_C(a)$ is represented uniquely by a degree-a polynomial on the plane: restricting the equation sequence gives this from $a < d$ and $H^1(\mathbb{P}^2, \mathcal{O}(a - d)) = 0$. There are $\binom{a+2}{2}$ such monomials. Summing from $a = 0$ to $d - 3$ gives $\binom{d}{3}$. Finally Y is affine and π is an isomorphism off the vertex. The space computed is the space of sections of the sheaf $R^1\pi_*\mathcal{O}_X$ supported there. For duality, plane adjunction and twisting sheaves see [Har77, Ch. III, §§5,7 and Ch. II, §8]. □

For $d = 3$ it is the constant section of $\omega_C = \mathcal{O}_C$ that contributes the sole obstruction. For $d = 4$, three independent linear forms and the constant section give four. Thus a single blow-up already resolves cones whose geometric genera can be arbitrarily large. This is why we need a second condition when we return to the question of which surface singularities are rational double points.

8. Valuations, divisors and the canonical class

8.1. **The order along a curve.** On a smooth complex curve near z_0 we write a nonzero meromorphic germ as $(z-z_0)^n u$ with u an invertible germ. Algebraically, let $(R,\mathfrak{m})$ be a regular local ring of dimension one. Choose a generator t of $\mathfrak{m}$. Every nonzero $b \in R$ is uniquely $t^m u$ for a unit u and $m \geq 0$: existence of the largest such m follows from Krull intersection; after dividing by t^m we have an element outside $\mathfrak{m}$. Applying this to numerator and denominator shows that every $a \in \mathrm{Frac}(R)^\times$ has a unique expression $t^n u$ with $n \in \mathbb{Z}$ (for this choice of t). A different generator is a unit times t, so $n = \mathrm{ord}_R(a)$ is independent of the choice. Orders add on products and $\mathrm{ord}_R(a+b) \geq \min(\mathrm{ord}_R a, \mathrm{ord}_R b)$, with $\mathrm{ord}_R(0) = +\infty$.

Lemma 8.1 (Arithmetic of orders)**.** *For a DVR R with fraction field K one has $R = \{h \in K : \mathrm{ord} h \geq 0\}$, using $\mathrm{ord} 0 = +\infty$. If two elements have different orders then the order of their sum is the smaller order.*

Proof. For nonzero h use its expression $\pi^n u$. If $n \geq 0$ it belongs to R. If $n < 0$ and h belonged to R, multiplying by π^{-n} would place the unit u in the maximal ideal. For the second assertion factor out the smaller power π^n. What remains is a unit plus an element of the maximal ideal, hence again a unit. When the two orders agree cancellation may occur: $(1+t)-1 = t$ in $k[t]_{(t)}$. Changing the uniformizer multiplies the coefficient of the lowest power by a unit, preserving the integer order. □

Exercise 8.2. Find the zeros and poles of $(x-a)^2/(x-b)^3$ on the entire projective line for $a \neq b$.

Solution. At a and b one reads 2 and -3 by using $x-a$ and $x-b$ as uniformizers. At infinity use $u = 1/x$. The function is $u(1-au)^2/(1-bu)^3$ and hence has order one. It is a unit at all other points. The total order is zero. This gives a first example where restricting to one affine chart hides a zero. □

On the affine line, if $h = (x-a)^r u(x)/(x-a)^s v(x)$ and $u(a)v(a) \neq 0$, this prescription gives $r-s$. On a normal surface take instead the local ring at the generic point of an irreducible curve D. It is one-dimensional and normal, hence a DVR. We obtain the integer $\mathrm{ord}_D h$ for each nonzero rational function h. For example along (x,z) on $xy = z^2$, the element y is a unit and z generates the maximal ideal, so $\mathrm{ord}_D x = 2$. This single integer will determine a relation in the divisor class group.

End of lecture 10

8.2. **Weil divisors, principal divisors and the class group.** Write $X^{(1)}$ for the codimension-one points of a normal integral Noetherian variety X. Each represents an irreducible divisor, its closure. On an affine open it corresponds to a height-one prime, a property unaffected by restricting to a smaller neighbourhood of that point. A *Weil divisor* is a finite sum $D = \sum n_P P$, $n_P \in \mathbb{Z}$. It is *effective* if all coefficients are nonnegative. For

instance $2V(y) - 3V(y^2 - x^3)$ is a divisor on the plane. The second prime divisor is the cusp of Figure 3; its coefficient records its multiplicity in this formal sum. For $a \in k(X)^\times$ define

$$\operatorname{div}_X(a) = \sum_{P \in X^{(1)}} \operatorname{ord}_P(a) P, \qquad \operatorname{Cl}(X) = \frac{\bigoplus_{P \in X^{(1)}} \mathbb{Z}P}{\{\operatorname{div}_X(a) : a \in k(X)^\times\}}.$$

On an affine chart, for $a = b/c$ the primes occurring lie among the height-one minimal primes of (b) and (c), of which there are finitely many. A finite affine cover proves finiteness globally. Orders add, so the principal divisors form a subgroup. Divisors with the same class are *linearly equivalent*, written $D \sim D'$.

Theorem 8.3 (Algebraic Hartogs principle)**.** *For a Noetherian normal domain A with fraction field K,*

$$A = \bigcap_{\operatorname{ht} \mathfrak{p} = 1} A_{\mathfrak{p}} \quad \text{inside } K.$$

Thus a rational function on $\operatorname{Spec} A$ *is regular precisely when its divisor is effective* [Sta26, Tag 0AVB].

Proof. Let $a/b \in K$ with $0 \neq b \in A$ and suppose $a \notin (b)$. The nonzero class $\bar{a}$ generates a nonzero submodule of $A/(b)$. Choose an associated prime $\mathfrak{p}$ of this submodule. Localizing there gives $\bar{a} \neq 0$ and $\operatorname{depth}(A/(b))_{\mathfrak{p}} = 0$. Since b is a nonzerodivisor, $\operatorname{depth} A_{\mathfrak{p}} = 1$. Condition (S_2) forces $\operatorname{ht} \mathfrak{p} \leq 1$, and $b \in \mathfrak{p}$ gives equality. Thus $a/b \notin A_{\mathfrak{p}}$. This proves the nontrivial inclusion. The depth facts used here are the regular-sequence criterion for depth and its change on quotienting by a nonzerodivisor [Sta26, Tag 00LE]. □

For example, a rational function regular on the punctured spectrum of a normal surface local ring is already in the ring. Its inverse has the same property if the function never vanishes there. In geometric language a single regular equation cannot cut out just the vertex on such a surface. If it had neither a zero nor a pole along any curve, both it and its inverse would extend to the vertex. The argument pinpoints what went wrong for the union of two planes at a point: the local ring there had depth one.

A height-one prime $\mathfrak{p}$ is a principal *ideal* exactly when its prime Weil divisor P is a principal *divisor*. Indeed, if $\mathfrak{p} = (a)$ then a generates $\mathfrak{p}A_{\mathfrak{p}}$ and is a unit at every other height-one prime, so $\operatorname{div}(a) = P$. Conversely if $\operatorname{div}(a) = P$, Hartogs gives $a \in A$ and its positive order gives $a \in \mathfrak{p}$. For $b \in \mathfrak{p}$ the divisor of b/a is effective when $b \neq 0$, whence $b/a \in A$. Thus $\mathfrak{p} = (a)$. Consequently A is factorial if and only if $\operatorname{Cl}(\operatorname{Spec} A) = 0$, using the height-one prime criterion for a Noetherian domain. In particular $\operatorname{Cl}(\mathbb{A}^n) = 0$.

Corollary 8.4 (Factorization and the class group)**.** *A Noetherian normal domain is factorial if and only if its Weil divisor class group vanishes.*

Proof. In a UFD every irreducible element defines a prime ideal. Conversely every height-one prime contains an irreducible factor of any nonzero element in it, hence is generated by that irreducible. It follows that every prime divisor is principal. Now suppose the class group is zero, and fix a height-one prime P. Choose $h \in K^\times$ with $\operatorname{div}(h) = P$. By the intersection formula above $h \in A$. For every $a \in P$ the divisor of a/h is effective, hence $a/h \in A$. Conversely $h \in P$, so $P = (h)$. Factorization into irreducibles exists by the ascending chain condition on principal ideals. If b is irreducible, choose a prime minimal over (b). It has height one, hence is (t) for a prime element t. Since t divides the irreducible b, these elements are associates. Thus all irreducibles are prime and factorization is unique. □

Lemma 8.5 (Removing a prime divisor)**.** *If P is a prime divisor on X and $U = X \setminus P$, there is an exact sequence*

$$\mathbb{Z}[P] \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0.$$

Proof. Every prime divisor on U extends by closure to X, and restricting a divisor erases precisely the term supported on P. The rational function fields agree. If the restriction of D is principal, subtract the divisor on X of the same function: only a multiple of P remains [Har77, II, Prop. 6.5]. □

Example 8.6 (The class group of a quadratic cone)**.** Let $A = k[x, y, z]/(xy - z^2)$, $P = V(x, z)$ and $Q = V(y, z)$. The complement of P is $D(x) = \operatorname{Spec} k[x, x^{-1}, z]$, which is factorial. Hence $[P]$ generates the class group. We computed $\operatorname{div}(x) = 2P$; similarly $\operatorname{div}(y) = 2Q$ and $\operatorname{div}(z) = P + Q$. To see that P itself is not principal, localize at the vertex $\mathfrak{m} = (x, y, z)$. The ideal (x, z) needs two generators: the classes of x, z in $(x, z)/\mathfrak{m}(x, z)$ are linearly independent since all terms of $\mathfrak{m}(x, z)$ have degree at least two. Therefore $\operatorname{Cl}(\operatorname{Spec} A) = \mathbb{Z}/2$, and the divisor also remains nonprincipal locally. For $xy = z^{n+1}$ the same calculation gives $\mathbb{Z}/(n + 1)$: all possible relations on P come from units of $k[x, x^{-1}, z]$, precisely the cx^j. This last argument also proves minimality of the relation for the quadric.

The analogous list for *completed complex* rational double points is as follows. Let L be the lattice on the exceptional curves, with intersection pairing. Resolution identifies the local class group with the discriminant group $L^\vee/L$ [Lip69, Lam86]. We will draw the curves in Section 9. This identifies precisely which lattice the following table concerns.

A_n	D_n, n even	D_n, n odd	E_6	E_7	E_8
$\mathbb{Z}/(n + 1)$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/4$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	0

For instance the A_n matrix has diagonal -2 and adjacent entries 1. Its cokernel relations express each successive basis class as a multiple of the first; the last relation is $(n + 1)e_1 = 0$. Its determinant has absolute value $n + 1$, by the same two-term recurrence.

Proposition 8.7 (Computing the discriminant groups)**.** *The finite groups in the preceding table are the cokernels of the indicated ADE Cartan matrices.*

Proof. The negative sign in the intersection form does not affect its cokernel. For a chain set $e_0 = e_{n+1} = 0$. Its relations $2e_i = e_{i-1} + e_{i+1}$ give $e_i = ie_1$ and $(n+1)e_1 = 0$. Conversely $e_i \mapsto i \bmod (n+1)$ satisfies every relation, so the generator has exact order $n+1$.

For D_n use the lattice $L = \{a \in \mathbb{Z}^n : \sum a_i \text{ is even}\}$ with its ordinary scalar product. The simple roots $e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n$ span L: differences move integer mass to the last coordinate and their span together with $2e_n$ accounts for every even sum. Their pairings give precisely the D_n matrix. The dual consists of integral vectors and vectors all of whose coordinates are half-integers. Indeed $v_i - v_j \in \mathbb{Z}$ and $v_i + v_j \in \mathbb{Z}$ are necessary and sufficient for pairing integrally with the roots. Modulo L the group has four elements generated by $[e_1]$ and $[h]$, $h = (1/2, \dots, 1/2)$. The former has order two, and $2h$ belongs to L exactly when n is even. For even n the two nonzero elements are independent; for odd n the class of h has order four.

For a three-armed graph let a, b, c be the arm lengths *plus one*. A chain of length $a-1$ has determinant a by the recurrence $d_j = 2d_{j-1} - d_{j-2}$, $d_0 = 1, d_1 = 2$. The end diagonal entry of its inverse is $(a-1)/a$ by cofactors. Eliminate the three chains against the central vertex. The determinant is

$$abc\left(2 - \frac{a-1}{a} - \frac{b-1}{b} - \frac{c-1}{c}\right) = abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1\right).$$

For $(2,3,3), (2,3,4), (2,3,5)$ the answers are $3, 2, 1$. A finite group of prime order is cyclic and a group of order one is trivial. This proves the three exceptional entries. The geometric identification of the completed rational surface class group with the discriminant group uses the resolution; see [Lip69, §§14–17]. □

Why the canonical divisor? Among all divisors, which one could one associate to a variety without first choosing a curve or a hyperplane? On a smooth surface a local generator is a two-form $du \wedge dv$. Under a change of coordinates it is multiplied by the Jacobian determinant. These determinants are the transition functions for a line bundle, $K_X = \det \Omega^1_X$. The bundle exists even when there is no nowhere-vanishing global form. The following calculation connects its sign with familiar projective geometry. On the affine chart of $\mathbb{P}^2$ take $\omega = dx \wedge dy$. Near a general point of the line at infinity write $u = 1/x$ and $v = y/x$. Then

$$dx = -u^{-2}du, \qquad dy = u^{-1}dv - vu^{-2}du, \qquad \omega = -u^{-3}du \wedge dv.$$

Thus $K_{\mathbb{P}^2} \sim -3H$. By adjunction, for a smooth surface $S_d \subset \mathbb{P}^3$ of degree d one has $K_{S_d} \simeq \mathcal{O}_{S_d}(d-4)$ [Har77, Ch. II, Example 8.20.3]. A smooth cubic has ample anticanonical bundle, a smooth quartic has trivial canonical

bundle, and for $d \geq 5$ the canonical bundle is ample. This is one reason cubic surfaces will return at the end of the course.

There is also a differential-geometric reading of this trichotomy over $\mathbb{C}$. In local holomorphic coordinates the Ricci form of a Kähler metric g is $-i\partial\bar{\partial}\log\det(g_{a\bar{b}})$. Its class divided by 2π is $c_1(T_X) = -c_1(K_X)$. The sign is important: it is $-K_X$ that is positive on a Fano manifold. On a compact Kähler manifold with $c_1(T_X) = 0$ in real cohomology, each Kähler class contains a unique Ricci-flat Kähler metric, by Yau's theorem [Yau78]. The algebraic calculation below needs no choice of metric. It extends the canonical class to normal surfaces by looking at orders of rational forms along prime divisors.

8.3. **Measuring the zeros and poles of a differential.** For a finitely generated extension K/k, the space $\Omega^1_{K/k}$ is generated over K by symbols da, additive in a, with $dc = 0$ for $c \in k$ and $d(ab) = a\,db + b\,da$. Put $\Omega^n_{K/k} = \bigwedge^n_K \Omega^1_{K/k}$. For $K = k(x_1, \ldots, x_n)$ a rational differential expands in the dx_i by the product and quotient rules. Independence follows by applying the partial derivations. If $K/k(x_1, \ldots, x_n)$ is finite separable, differentiating a separable minimal polynomial expresses the differential of an algebraic generator in terms of the same dx_i. It also extends each derivation uniquely. Thus the dx_i are a basis for any separating transcendence basis, and the space of top forms is a line. Over a perfect field every finitely generated field extension admits such a basis. In characteristic p, $d(t^p) = 0$ illustrates the need for separability.

Lemma 8.8 (Differentiating through a finite extension)**.** *If $t_1, \ldots, t_n$ is a separating transcendence basis of K/k, then $dt_1, \ldots, dt_n$ form a K-basis of $\Omega^1_{K/k}$.*

Proof. Write $F = k(t_1, \ldots, t_n)$. Differentiation of rational functions and the partial derivatives $\partial/\partial t_i$ prove the statement for F. In a simple separable extension $F(a)$ the relation $P(a) = 0$ gives $P'(a)\,da = -\sum a^j\,dc_j$, where $P(T) = \sum c_j T^j$ is the minimal polynomial. The nonzero coefficient $P'(a)$ determines da uniquely in terms of the old differentials. Conversely every k-derivation of F with values in K extends by this very formula: the rule on a annihilates the relation and descends to the quotient by P. Extending the partial derivatives proves linear independence. A finite separable extension admits a primitive element, or one may repeat this argument for a finite list of generators. □

In characteristic p, $k(t)$ over $k(t^p)$ shows why one specifies *separating*. Relative to k, $d(t^p) = pt^{p-1}dt = 0$, while $dt \neq 0$ as the derivation $\partial/\partial t$ proves. The zero on the left gives no equation from which to solve for dt. More geometrically, the coordinate t^p still distinguishes the generic point algebraically, but it has lost its first-order variation. The very same

phenomenon explains the vanishing of a candidate top form in characteristic two on the quadratic cone.

Let X be normal, of finite type and dimension n over a perfect field k. At a codimension-one point P, the ring $\mathcal{O}_{X,P}$ is regular and lies on the smooth locus. Choose a uniformizer t and elements $u_2, \dots, u_n$ lifting a separating transcendence basis of $\kappa(P)/k$. Then $dt, du_2, \dots, du_n$ are a basis of the localized differentials: modulo the maximal ideal the last $n-1$ forms give a basis of the residue-field differentials and dt gives the conormal direction. Nakayama and smoothness give the assertion. Thus $dt \wedge du_2 \wedge \cdots \wedge du_n$ generates the lattice of regular top forms at P. Another such basis changes the generator by a unit. Writing a nonzero rational top form as a times this generator defines $\operatorname{ord}_P(\omega) = \operatorname{ord}_P(a)$ independently of these choices.

The divisor $K_X = \operatorname{div}_X(\omega) = \sum_P \operatorname{ord}_P(\omega) P$ is a *canonical divisor*. Multiplying by a rational function changes it by a principal divisor, so its class is intrinsic. On the smooth locus this describes the top exterior power of the cotangent bundle. The familiar change-of-variables determinant is exactly the transition function for this line bundle. On a normal surface, extending it as a reflexive sheaf gives the canonical sheaf. This is the class that enters adjunction and duality, and whose pullback will let us compare the singular surface with its resolution.

8.4. **Three canonical divisors.** On $\mathbb{A}^2$ the form $dx \wedge dy$ is everywhere a generator, hence has divisor zero. On $xy = z^2$ in characteristic different from two, differentiating the equation gives $y\,dx + x\,dy = 2z\,dz$. On $D(x)$ and $D(y)$ respectively,

$$dx \wedge dy = \frac{2z}{x}\, dx \wedge dz = -\frac{2z}{y}\, dy \wedge dz.$$

The first expression vanishes to order one on Q, the second to order one on P. Together these two opens cover every codimension-one point. The divisor is $P + Q = \operatorname{div}(z)$, so the canonical class is zero even though the class group is not. In characteristic two $dx \wedge dy = 0$, but $dx \wedge dz/x = -dy \wedge dz/y$ on the overlap is a rational generator without divisorial zeros or poles in every characteristic.

Exercise 8.9 (Why the canonical class measures change of coordinates)**.** On $\mathbb{P}^2$ with affine coordinates $x = X/Z$ and $y = Y/Z$, compute the order of $dx \wedge dy$ along the line at infinity.

Solution. At the generic point of that line, $X \neq 0$. Put $u = Z/X$ and $v = Y/X$, so $x = 1/u$ and $y = v/u$. Then

$$dx = -u^{-2}du, \quad dy = u^{-1}dv - vu^{-2}du, \qquad dx \wedge dy = -u^{-3}\, du \wedge dv.$$

It has a pole of order three. There are no other zeros or poles because it is a basis on $\mathbb{A}^2$. Hence $K_{\mathbb{P}^2} \sim -3H$ for a line H. The derivative determinant is precisely the transition factor of the line bundle of top forms. This same

line bundle appears in adjunction, where $(K_S + C)|_C = K_C$ for a smooth curve C on a smooth surface S, and in duality, where it pairs $H^i(S, L)$ with $H^{2-i}(S, L^{-1} \otimes \omega_S)$ on a smooth projective surface [Har77, Ch. II, §8 and Ch. III, §7]. □

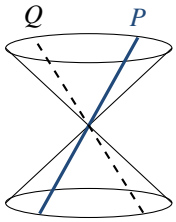


FIGURE 18. Two ruling lines on the quadric cone. Setting $z = 0$ gives the reduced union $P + Q$; setting $x = 0$ gives $2P$. The multiplicity in the second assertion is read at the generic point of the line.

For a different outcome take the affine cone over the twisted cubic

$$A = k[x, y, z, w]/(xz - y^2, xw - yz, yw - z^2) \simeq k[s^3, s^2t, st^2, t^3].$$

The parametrization identifies A with the third Veronese subring, which is normal: its monomials form the intersection of the nonnegative quadrant with the lattice $i + j \equiv 0 \bmod 3$, a saturated semigroup in that lattice. Here is a direct argument which also applies when the characteristic divides the Veronese degree. In $S = k[s, t]$ collect terms by total degree modulo three. The component of residue degree zero is A. An element u of $\mathrm{Frac}(A)$ integral over A satisfies a monic polynomial over S and lies in $\mathrm{Frac}(S)$; normality of the polynomial ring gives $u \in S$. Write $u = a/b$ with $a, b \in A$ and $b \neq 0$. Multiplication by b preserves each of the three residue degrees and kills no nonzero component. Since $bu = a$ has only residue degree zero, so does u, yielding $u \in A$. Write $P = V(y, z, w)$ and $Q = V(x, y, z)$. The chart $D(x)$ has coordinates $x^{\pm 1}, y$, with $z = y^2/x$, $w = y^3/x^2$. Symmetrically $D(w)$ has coordinates $w^{\pm 1}, z$. The intermediate charts $D(y)$ and $D(z)$ lie in their intersection. These opens cover the punctured cone. In $D(x)$ the uniformizer along P is y, and in $D(w)$ the uniformizer along Q is z. It follows that

$$\mathrm{div}(w) = 3P, \quad \mathrm{div}(x) = 3Q, \qquad \mathrm{div}(y) = P + 2Q.$$

Deleting Q leaves the factorial chart $D(x)$, whose only units are cx^j. Thus the full group of relations on Q is generated by $3Q$ and $\mathrm{Cl}(A) = \mathbb{Z}/3$. The local class also has exact order three: the ideal (x, y, z) needs three generators at the vertex by its independent linear terms. Already $[P] = [Q]$ by the displayed relations.

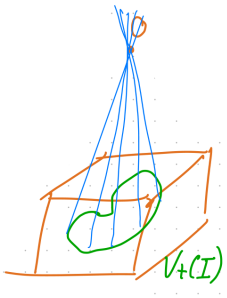


FIGURE 19. The cone on the twisted cubic from the eleventh lecture. The green curve denotes the projective image of $[s:t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Each of its points labels a blue ruling of the affine cone in $\mathbb{A}^4$. The two rulings $t = 0$ and $s = 0$ give the prime divisors P and Q in the calculation.

Lemma 8.10 (The equations and the divisors of the twisted cone)**.** *The three quadrics in the presentation above generate the full homogeneous ideal of the parametrization. The only prime divisor outside $D(x)$ is Q.*

Proof. Index the four generators by $0, 1, 2, 3$, representing $s^{3-i}t^i$. The three relations replace a pair of indices differing by at least two by the two closest indices of the same sum: $0 + 2 \rightsquigarrow 1 + 1$, $0 + 3 \rightsquigarrow 1 + 2$, $1 + 3 \rightsquigarrow 2 + 2$. Each step strictly decreases the sum of the squares of the indices. In a product of m generators, all final indices differ by at most one, and their multiset is uniquely determined by m and their sum j. The image of this normal monomial is $s^{3m-j}t^j$. Different values of j, and different degrees m, are linearly independent. Thus the kernel consists exactly of the stated quadrics.

Modulo x the relations give $y^2 = 0$ and then $z^2 = yw$, so every prime containing x contains y and z. Conversely (x, y, z) has quotient $k[w]$, hence is prime of height one. A prime divisor containing x must be this one. On $D(w)$ the equation $yw = z^2$ gives $y = z^2/w$, and $xw = yz$ gives $x = z^3/w^2$. Since w is a unit at the generic point of Q, one reads off orders 2 and 3, respectively. On $D(x)$ exchange the roles of s, t and one gets the analogous computation for P. In particular the rational function y vanishes once on P and twice on Q. No other prime divisor occurs, since y is invertible wherever both x and w are invertible. The intermediate open $D(y)$ is already such an open: $xz = y^2$ forces both x and z to be units there, then $yw = z^2$ forces w to be a unit too. □

It is worth spelling out what the order-three relation says about ideals. Write $\mathfrak{q} = (x, y, z)$ for the ideal of Q. The equality $\operatorname{div}(x) = 3Q$ gives

$$\mathfrak{q}^{(3)} = \mathfrak{q}^3 A_{\mathfrak{q}} \cap A = (x).$$

Indeed $a \in A$ lies on the left precisely when $\operatorname{ord}_Q(a) \geq 3$. Then a/x has nonnegative order at Q and at every other prime divisor, so $a/x \in A$ by

Hartogs. The reverse inclusion follows at once from $\operatorname{ord}_Q(x) = 3$. The ordinary power $\mathfrak{q}^3$ is strictly smaller: all its homogeneous elements have degree at least three, whereas x has degree one in the coordinate ring of the cone. Interchanging x, y, z, w with w, z, y, x likewise gives $(y, z, w)^{(3)} = (w)$. The divisor group records these orders at the generic points of curves; it cannot detect this additional difference of ideals supported at the vertex.

For $\operatorname{char} k \neq 3$, compute

$$dx \wedge dw = \frac{3y^2}{x^2}\, dx \wedge dy \quad \text{on } D(x).$$

Interchanging (x, y) with (w, z) changes the form by a sign. Its divisor is therefore $2P + 2Q$. Dividing by y leaves the representative P of K_X. This gives canonical index *three*. A computation valid also in characteristic three uses $\eta = dx \wedge d(y/x)$ on $D(x)$. Put $u = z/w$, so that $x = wu^3$ and $y/x = 1/u$. Then $\eta = -u\, dw \wedge du$ on $D(w)$, hence $\operatorname{div}(\eta) = Q$ in every characteristic.

Proposition 8.11 (An independent index computation)**.** *For the affine cone over the d-uple embedding of $\mathbb{P}^1$, the canonical class at the vertex has index $d/\gcd(d, 2)$, over an algebraically closed field in any characteristic.*

Proof. Let $x = s^d$ and $w = t^d$ and let P denote $t = 0$ and Q denote $s = 0$ as prime divisors of the cone. On $D(x)$ the coordinates are $x^{\pm 1}, u = t/s$, and on $D(w)$ they are $w^{\pm 1}, v = s/t$. The two charts cover the punctured cone, with $w = xu^d$ and $v = u^{-1}$. Restricting divisors to the factorial open $D(x)$ shows that the class of Q generates the global class group. A relation supported on Q is the divisor of a unit on $D(x)$, hence the divisor of cx^j. Since $x = wv^d$ near Q, the relations are exactly $d\mathbb{Z}[Q]$.

To check that passage to the local ring at the vertex does not shorten the order, let $1 \leq j < d$. The ideal of functions with $\operatorname{ord}_Q \geq j$ is generated by the degree-one monomials $s^d t^0, s^{d-1}t, \ldots, s^j t^{d-j}$. Every higher monomial of total degree a multiple of d and s-exponent at least j is divisible by one of these *in the Veronese ring*: choose the first degree-d factor using at least j of its s's and at most the available number of t's. The listed generators are independent modulo the vertex ideal times this ideal, by degree. Thus at least $d - j + 1 > 1$ generators are needed even after localizing at the vertex. If jQ were Cartier there, its ideal would be principal. Together with $\operatorname{div}(x) = dQ$ this proves exact order d locally.

Now take the top form $\omega = (dx/x) \wedge du$. It generates on $D(x)$. On $D(w)$ use $x = wv^d$ and $u = v^{-1}$:

$$\omega = \left(\frac{dw}{w} + d\,\frac{dv}{v}\right) \wedge (-v^{-2}dv) = -v^{-2}\frac{dw}{w} \wedge dv.$$

Here the first d in $d\,dv/v$ is the integer degree. Its value in k has no effect on the wedge product. Hence $\operatorname{div}(\omega) = -2Q$. The order of -2 in $\mathbb{Z}/d\mathbb{Z}$ gives the result. For the twisted cubic $d = 3$ this gives index three; for the conic $d = 2$ it gives index one. It also locates the source of the two different

integers in the twisted-cubic calculation: 2 is a coefficient of a canonical representative, whereas 3 is the order of its class. □

A normal local surface is Cohen–Macaulay by (S_2). It is *Gorenstein* precisely when its canonical module is free of rank one, equivalently K_X is Cartier. A Cartier divisor on a local scheme is principal. It is $\mathbb{Q}$-*Gorenstein* if some positive multiple of K_X is Cartier; the least such multiple is its index. On a general variety these are local conditions: Cartier does not require a single global equation.

Proposition 8.12. *A local complete intersection is Gorenstein. In particular a normal hypersurface has Cartier canonical class.*

Proof. Let $A = R/(f_1, \dots, f_c)$ be the quotient of a regular local ring by a regular sequence. The Koszul complex resolves A. Its dual is the same complex in reversed degrees, so $\operatorname{Ext}^c_R(A, R) = A$. This is the canonical module (choose a generator of the canonical module of R); thus it is free of rank one. For a single equation the entire calculation is the dual of $0 \to R \xrightarrow{f} R \to A \to 0$. For the Gorenstein property under local complete intersections see [Sta26, Tag 0DWA]. □

Exercise 8.13 (Detecting Gorensteinness in a zero-dimensional section)**.** Compare the vertices of the quadric cone $z^2 = xy$ and the cone on the twisted cubic. In the first local ring kill x, y, and in the second kill x, w. What distinguishes the resulting rings? Can the calculation recover the Gorenstein property of the two surface points?

Solution. Both local rings are normal surfaces, hence Cohen–Macaulay. The displayed pairs are systems of parameters, since each quotient below is Artinian; they are therefore regular sequences. In the quadric case one obtains $B_2 = k[z]/(z^2)$. In the cubic case the three equations become $y^2 = yz = z^2 = 0$, so one obtains

$$B_3 = k[y, z]/(y^2, yz, z^2).$$

The *socle* of a local Artinian ring $(B, \mathfrak{n})$ is $\{b : \mathfrak{n}b = 0\}$. It is kz in B_2 and $ky \oplus kz$ in B_3.

Here is why the dimensions decide the question. For a finite-dimensional local k-algebra with residue field k, the canonical module is $D(B) = \operatorname{Hom}_k(B, k)$, with $(b\ell)(a) = \ell(ba)$. Under the evaluation pairing with B, the annihilator of $\mathfrak{n}D(B)$ is exactly $\operatorname{soc}(B)$. Thus $\dim_k D(B)/\mathfrak{n}D(B) = \dim_k \operatorname{soc}(B)$. By Nakayama, $D(B)$ is cyclic precisely when the socle is one-dimensional. A surjection $B \to D(B)$ is then an isomorphism since both sides have the same k-dimension. This gives the Artinian Gorenstein criterion; for Artinian duality see [Sta26, Tag 08YW]. Gorensteinness is preserved and reflected upon quotient by a regular element of the maximal ideal [Sta26, Tag 0BJJ]. Apply this twice. The quadric vertex is Gorenstein, whereas the twisted cubic cone is not. In fact its type is two:

present the Cohen–Macaulay ring R as a quotient of a regular local ring S in codimension c. Its canonical module is $\mathrm{Ext}^c_S(R,S)$ and its higher Ext groups against S vanish, by a length-c free resolution. Applying $\mathrm{Ext}_S(-,S)$ to $0 \to R \xrightarrow{f} R \to R/(f) \to 0$ gives $\omega_{R/(f)} = \omega_R/f\omega_R$. Applying this to the two parameters shows that the minimal number of canonical generators is the socle dimension computed above. The divisor calculation gives the further information that its canonical index is three. □

This recovers the result of the explicit top-form calculation for the quadratic cone.

End of lecture 11

9. What the exceptional curves remember

9.1. **Pulling back the canonical divisor.** Let $\pi : X \to Y$ be a resolution of a normal surface germ over an algebraically closed field. A prime divisor E on X is *exceptional* if its image is a point. Pullback of a principal divisor is particularly easy: the two surfaces have the same function field, and we take the divisor of the same rational function on X. Exceptional orders can be nonzero. On the blow-up of the origin of $\mathbb{A}^2$, $x = x$ on one chart and $x = sy$ on the other, so the total transform of $V(x)$ consists of its strict transform and the exceptional line, both with coefficient one. For Cartier divisors the same construction works locally and glues since the change of local equation is a unit. If mD is Cartier with $m > 0$, set $\pi^*D = \pi^*(mD)/m$; taking a common multiple shows independence of m.

Suppose Y is $\mathbb{Q}$-Gorenstein. Choose the *same* rational top form to define K_X and K_Y. Off the exceptional set the divisors agree, so there are unique rational numbers a_i such that

$$K_X = \pi^* K_Y + \sum_i a_i E_i.$$

Multiplication of the form by a rational function changes the first two terms by matching principal divisors, so the coefficients are independent of the form. They are called *discrepancies*. A singularity, with empty boundary here, is *terminal*, *canonical*, *Kawamata log terminal* (klt), or *log canonical* according as the discrepancy of every exceptional prime divisor over Y is > 0, ≥ 0, > -1, or ≥ -1 [KM98, Ch. 4]. The quantifier includes divisors on higher resolutions. For a log resolution the inequalities can be checked there (for terminality also use the smooth blow-up formula below). The terminology comes from the minimal model program: the comparison of canonical classes controls which singularities can arise under birational contractions.

For the blow-up of a smooth point, $dx \wedge d(xt) = x\,dx \wedge dt$, so $K_{X'} = \sigma^* K_X + F$ with F exceptional. Blowing up a point on a simple normal crossings exceptional divisor produces a new discrepancy $1 + \sum_{i : P \in E_i} a_i$; at most two components pass through such a point. This calculation explains

why a zero discrepancy on the minimal resolution is compatible with positive discrepancies above it.

9.2. **Intersections, minimality and the fundamental cycle.** Write $L = \bigoplus_i \mathbb{Z}E_i$. Intersection on a smooth surface defines a symmetric integral pairing on L. For distinct curves meeting properly it is the sum of their local intersection lengths; for smooth curves meeting transversely it counts their common points. The self-intersection of E is the degree of its normal bundle when E is smooth. The pairing is compatible with further blow-ups: total pullback preserves intersections, and is orthogonal to the new exceptional curve of square -1. The intersection matrix of all exceptional curves over one normal surface point is *negative definite*; this is the negativity theorem for surface resolutions [Mum61, Art66]. Its support is connected by proper birationality and normality. We may arrange that the reduced divisor has simple normal crossings by further blow-ups if necessary.

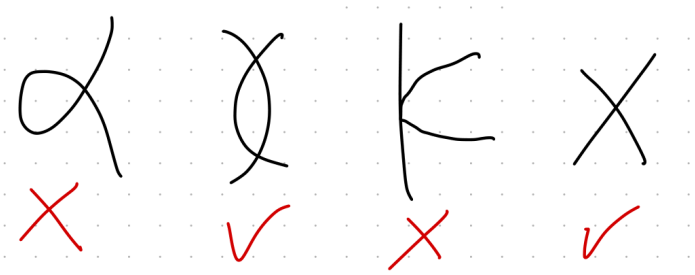


FIGURE 20. Testing for simple normal crossings in the last lecture. From the left: one component crosses itself, two smooth components meet transversely twice, two smooth components are tangent, and two smooth components meet transversely once. The second and fourth pictures satisfy the local crossing condition with smooth components.

Exercise 9.1 (Counting intersections in the four sketches)**.** Why does meeting twice cause no difficulty for simple normal crossings? Can two exceptional (-2)-curves above a normal surface point meet as in the second sketch?

Solution. On a smooth surface a reduced divisor has simple normal crossings when its irreducible components are smooth and at each point it is empty, a single parameter axis, or the union of the two axes in regular local parameters. This is checked at each point separately. Two transverse intersection points each contribute one to $E \cdot F$. In a tangency, for instance $y = 0$ and $y = x^2$, the local intersection length is instead $\dim_k k[[x]]/(x^2) = 2$ at a single point. A self-crossing fails the smoothness requirement on the individual component. Two exceptional (-2)-curves meeting twice would have $(E + F)^2 = -2 + 2(2) - 2 = 0$, contradicting negative definiteness. Thus

the second sketch is allowed for a general simple normal crossings divisor but is excluded in an ADE exceptional fibre. □

A smooth rational curve of self-intersection -1 on a smooth surface contracts to a smooth point (Castelnuovo's criterion). An exceptional such curve can therefore be removed while keeping a resolution. Repeated contraction gives the unique minimal resolution, characterized by having no exceptional (-1)-curve; any resolution factors uniquely through it. The distinction between a *minimal resolution* and a *minimal resolution with simple normal crossings* matters in general [Sta26, Resolution of Surfaces, §§16–17]. For rational double points the minimal resolution already has simple normal crossings.

Adjunction for an integral projective curve on X gives

$$K_X \cdot E + E^2 = 2p_a(E) - 2, \qquad p_a(E) = 1 - \chi(\mathcal{O}_E).$$

In the smooth case $p_a(E)$ is its genus. Also $\pi^* K_Y \cdot E_i = 0$, by the projection formula for a Cartier multiple. Hence *one can solve for discrepancies*:

$$\sum_j a_j (E_j \cdot E_i) = 2p_a(E_i) - 2 - E_i^2.$$

The matrix is nonsingular. On the minimal resolution each right-hand side is nonnegative, since the only negative possibility would be $p_a = 0, E^2 = -1$, a smooth rational (-1)-curve. If Y is canonical then $D = \sum a_i E_i$ is effective, and $D^2 = \sum_i a_i (K_X \cdot E_i) \geq 0$. Negativity forces $D = 0$. Thus the minimal resolution of a singular canonical surface is crepant. (Crepant means $K_X = \pi^* K_Y$.) Terminal surfaces are therefore smooth. Conversely a smooth surface is terminal, by successive applications of the smooth blow-up formula.

Lemma 9.2 (Intersection with a proper exceptional curve)**.** *For an exceptional integral curve E on a smooth surface X, the rule $D \cdot E = \deg_E \mathcal{O}_X(D)|_E$ depends only on the linear equivalence class of D. If D is pulled back from a Cartier divisor on the base of the resolution, then $D \cdot E = 0$.*

Proof. A principal divisor defines the trivial line bundle. Tensor product adds degrees on the proper curve, proving invariance and additivity. For a pullback, the map from E to the base factors through the closed point. A line bundle from that point pulls back to the constant line bundle on E. If a Cartier divisor consists of another curve F meeting E properly, its defining section restricts to a section on E with zeros precisely on $E \cap F$. The sum of the local lengths is its degree. For $F = E$ the restriction is its normal line bundle, so self-intersection still has a meaning when there is no second curve to intersect. These constructions and the projection formula are developed in [Har77, Ch. V, §1]. □

Exercise 9.3 (Following a principal divisor through a blow-up)**.** At the origin of the plane, pull back x, y, and x^2+y^3. What changes if one chooses a different rational top form to define the discrepancies?

Solution. On the x-chart one has $x = x$, $y = xt$, and $x^2+y^3 = x^2(1+xt^3)$. At the generic point of the exceptional line x is a uniformizer, t is a unit, and the last parenthesis is a unit. The orders are therefore $1, 1, 2$. A function from downstairs can have a nonzero exceptional order. If the same top form on the common function field is replaced by $h\omega$, the divisors upstairs and downstairs change by $\mathrm{div}_X(h)$ and $\mathrm{div}_Y(h)$. Pullback of the second equals the first *including* these exceptional orders. Subtracting leaves every discrepancy unchanged. □

Example 9.4 (Three cones revisited)**.** For a cone whose resolution has a single exceptional curve E of genus g with $E^2 = -d$, the discrepancy, provided the canonical divisor is $\mathbb{Q}$-Cartier, satisfies $-da = 2g-2+d$. For a conic this gives $a = 0$. For a smooth plane cubic it gives $a = -1$. For the twisted cubic the exceptional curve is $\mathbb{P}^1$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-3)$, so $a = -1/3$, just as the index-three calculation on the punctured cone allows. The three cones are respectively canonical, log canonical but not klt, and klt but not canonical. In the cubic case the nonzero $H^1(E, \mathcal{O}_E)$ computed on the resolution is another way of seeing the difference.

Definition 9.5 (Fundamental cycle)**.** The fundamental cycle on a resolution over one singular point is the componentwise least nonzero effective exceptional cycle Z satisfying $Z \cdot E_i \le 0$ for all i. It is supported on the whole fibre.

Here is a practical construction. Start at the reduced exceptional cycle. If $D \cdot E_i > 0$, add E_i and repeat. There exists a strictly positive integral cycle B with all $B \cdot E_i < 0$: the negative intersection matrix is positive definite with nonpositive off-diagonal entries, so its inverse has nonnegative entries (solve against the vector of ones and split the solution into its positive and negative parts). After clearing denominators and multiplying, B bounds the starting cycle. The procedure stays below any effective nonzero cycle A with $A \cdot E_i \le 0$. Indeed such an A has full support by connectedness, and if $D_i = A_i$ then $D \cdot E_i \le A \cdot E_i$ since all off-diagonal intersections are nonnegative. Thus we never increase a coefficient already at the bound. Termination follows from B, and the same argument against every A proves minimality and uniqueness.

9.3. **Rationality and the ADE characterization.** A normal surface singularity is *rational* if $R^1\pi_*\mathcal{O}_X = 0$ for a resolution, equivalently for every resolution. This condition measures the cohomology over the point. It is independent of resolution since a blow-up at a smooth point has vanishing higher direct image of its structure sheaf, and resolutions admit a common resolution. Over an affine base the vanishing is equivalent to $H^1(X, \mathcal{O}_X) = 0$.

In particular this is a local condition on Y: it does not assert that the projective surface carrying the point is birational to a plane.

What does the higher direct image measure? The functor π_* on sheaves is left exact. Given an injective resolution $\mathcal{O}_X \to I^\bullet$, take $\pi_* I^\bullet$ term by term; its degree-one cohomology sheaf is $R^1\pi_*\mathcal{O}_X$. There is a concrete way to read the definition. It is the sheaf associated with $U \mapsto H^1(\pi^{-1}U, \mathcal{O}_X)$ [Har77, Ch. III, Proposition 8.1]. On an open where π is an isomorphism this sheaf vanishes. For affine Y, higher cohomology of the quasi-coherent sheaves $R^q\pi_*\mathcal{O}_X$ vanishes, so the Leray sequence identifies $H^1(X, \mathcal{O}_X)$ with $H^0(Y, R^1\pi_*\mathcal{O}_X)$. For the elliptic cone a class in $H^1(C, \mathcal{O}_C)$ survives above every neighbourhood of the vertex. For the rational conic cone the cover by the two affine line-bundle charts left no such class. The definition packages precisely this difference in behaviour under contraction.

For an effective exceptional cycle the adjunction formula reads $2p_a(Z) - 2 = Z^2 + K_X \cdot Z$. Artin's criterion says that the singularity is rational exactly when $p_a(Z) = 0$ for the fundamental cycle. Equivalently $p_a(D) \le 0$ for all nonzero effective exceptional cycles D [Art66, Lip69]. For a rational singularity one further has

$$e(\mathcal{O}_{Y,y}) = -Z^2, \qquad \operatorname{edim}(\mathcal{O}_{Y,y}) = e(\mathcal{O}_{Y,y}) + 1.$$

Here e is the Hilbert–Samuel multiplicity of the maximal ideal: in dimension two $\operatorname{length}(A/\mathfrak{m}^r) = e\,r^2/2 + O(r)$ for large r. For rational surface singularities the maximal ideal becomes $\mathcal{O}_X(-Z)$ on the minimal resolution, and its colengths are computed by cycles there [Art66, Lip69]. Without rationality one cannot identify this pullback with the fundamental cycle in general.

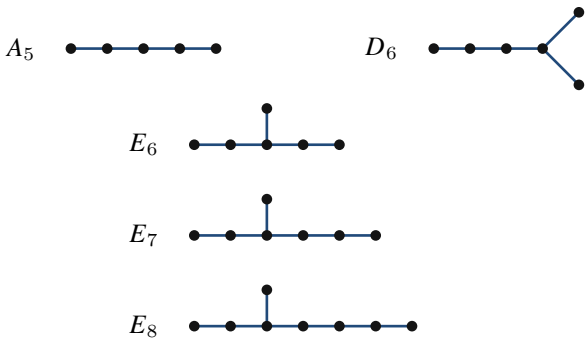
Theorem 9.6 (Rational double points)**.** *Let y be a singular closed point of a normal algebraic surface over an algebraically closed field. Each of the following conditions characterizes a rational double point.*

1. *The singularity is rational of multiplicity two.*
2. *It is rational of embedding dimension three.*
3. *It is a hypersurface double point which can be resolved by successively blowing up closed points alone (an absolutely isolated double point).*
4. *It is rational and Gorenstein.*
5. *It is canonical.*
6. *Its minimal exceptional divisor consists of smooth rational curves of self-intersection -2, with intersection graph a finite Dynkin diagram A_n $(n \ge 1)$, D_n $(n \ge 4)$, E_6, E_7 or E_8.*
7. *The completed local ring has one of the ADE equations, using Artin's characteristic-dependent forms when necessary.*

Over $\mathbb{C}$ one can add that the completed germ is the quotient of a smooth plane germ by a nontrivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. See [DV34, Art66, Lip69, Art77, Dur79] *and* [Sta26, Tag 0BGB].

Proof. Artin's multiplicity formulas give (1)⇔(2). Embedding dimension three gives a hypersurface by the complete local presentation, hence Gorenstein. The converse implication from rational Gorenstein to double is the Gorenstein rational surface theorem [Sta26, Tag 0BGB]. Lipman's results on point blow-ups of rational singularities and the absolutely isolated double point criterion give (3); the cone over a smooth plane cubic shows why the word double matters [Dur79, §4]. For the absolutely isolated criterion in arbitrary characteristic see also [Roc96, §1]. From (5) the minimal resolution is crepant by the argument above. Adjunction now gives $E_i^2 = 2p_a(E_i) - 2 < 0$, forcing $p_a(E_i) = 0$ and $E_i^2 = -2$. An integral curve of arithmetic genus zero over an algebraically closed field is a smooth $\mathbb{P}^1$, by the normalization sequence. Two of these curves cannot meet twice: the square of their sum would be nonnegative. A cycle in the graph is excluded for the same reason, as is a triple intersection of distinct curves.

More explicitly, the negative of the intersection matrix now has 2 on the diagonal and -1 for adjacent vertices. Positive definiteness gives exactly the finite simply laced diagrams. A vertex of degree four would contain an affine D_4 subdiagram (put coefficient two at the centre and one on the neighbours to get square zero). Two trivalent vertices likewise give an affine D subdiagram by taking the path between them with coefficient two and the four terminal neighbours with coefficient one. A tree with at most one trivalent vertex is a chain or has three arms. For arm lengths $a-1, b-1, c-1$, a Schur complement at the centre gives $1/a + 1/b + 1/c > 1$. This is *the same inequality* as in the calculation on $\mathbb{P}^1$. It gives $(2, 2, n-2)$ for D_n and $(2, 3, 3), (2, 3, 4), (2, 3, 5)$ for the exceptional diagrams.



The first row shows a chain and a fork, with representative sizes.
The three exceptional graphs have respectively six, seven and eight vertices.

Conversely, for an ADE configuration, adjunction gives $K_X \cdot E_i = 0$. Starting at any single component and adding a component whenever the intersection is positive computes the fundamental cycle by the bounding argument above (connectedness forces the eventual support to be full). At each step the current cycle has square -2. A positive intersection must be exactly one: the square after adding a (-2)-curve would otherwise be nonnegative.

Thus the square remains -2 all the way to Z. Adjunction and Artin's criterion yield $p_a(Z) = 0$ and multiplicity two. This proves (6)⇒(1) without presupposing that the original canonical class was Cartier. Conversely a rational Gorenstein double point has $K_X \cdot Z = 0$ by adjunction, and each $K_X \cdot E_i \geq 0$ on the minimal resolution, so all vanish. Since K_Y is Cartier and the matrix is nonsingular, the discrepancies vanish as well. Repeated smooth blow-ups give canonicity. □

Proposition 9.7 (Seeing rationality on successive exceptional cycles)**.** *Let D be an effective exceptional cycle on a smooth surface and E a smooth rational exceptional component. There is an exact sequence*

$$0 \longrightarrow \mathcal{O}_E(-D) \longrightarrow \mathcal{O}_{D+E} \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

If $D \cdot E = 1$ then the restriction induces isomorphisms on both H^0 and H^1.

Proof. The ideal of $D + E$ is $\mathcal{O}_X(-D - E)$; quotient the ideal $\mathcal{O}_X(-D)$ by it. The resulting sheaf is $\mathcal{O}_E(-D)$, the claimed kernel. When $D \cdot E = 1$, it is a line bundle of degree -1 on $\mathbb{P}^1$. Its two cohomology groups vanish. This gives the assertion even if D already contains E. Thus the scheme structure of repeated components matters, and still admits computation one component at a time.

More generally $\chi(\mathcal{O}_E(-D)) = 1 - D \cdot E$. Additivity yields $p_a(D + E) = p_a(D) + D \cdot E - 1$. On an ADE configuration, if one begins at a single vertex, $D^2 = -2$. A positive $D \cdot E$ must equal one, since $(D + E)^2 = -4 + 2D \cdot E$ has to stay negative. Inductively the square remains -2 and $H^1(\mathcal{O}_D) = 0$ throughout the computation sequence. Its end is the fundamental cycle, so Artin's criterion applies. For a general resolution, the theorem on formal functions computes the completion of $R^1\pi_*\mathcal{O}_X$ from the cohomologies of the infinitesimal inverse images of the closed point [Har77, Ch. III, Theorem 11.1]. This explains the appearance of exceptional *cycles* as opposed to just a topological graph of the reduced fibre. □

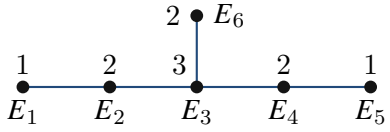
Exercise 9.8 (Why only five shapes of diagram occur)**.** Classify connected simply laced positive definite diagrams all of whose diagonal entries are 2.

Solution. On any cycle, placing coefficient one at its vertices gives square at most zero for the Cartan form. No cycle occurs. A vertex with four neighbours gives a nonpositive vector by placing 2 at the centre and 1 on the four neighbours. If there were two branching vertices, put 2 on the path between them and 1 on the four first off-path neighbours; its square is again zero. These are direct tests on principal submatrices. What remains is a chain, or a tree with one trivalent vertex. In the latter case let $2 \leq a \leq b \leq c$ be the arm lengths plus one. All chain blocks are positive definite. The elimination calculation for the discriminant in the preceding lecture gives

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1.$$

If $a \geq 3$ the sum is at most one. For $a = 2$ there is the family $(2, 2, c)$, and for $b = 3$ there are exactly $(2, 3, 3), (2, 3, 4), (2, 3, 5)$. For $b \geq 4$ the sum is again at most one. We get $A_n, D_{c+2}, E_6, E_7, E_8$. The inequality is the one obtained by counting orbits of rotations in the first lecture. This time it has arisen as positivity of an intersection matrix on a resolution. □

9.4. **The six curves of E_6.** Number the five curves in the chain from left to right and attach E_6 to E_3. In the following diagram the number above or beside a vertex is its coefficient in the fundamental cycle, and every vertex represents a curve of square -2.



Starting at $(1, 1, 1, 1, 1, 1)$ the only positive intersection is at E_3. Add E_3, then E_2, E_4, then E_3 again, and finally E_6. Every addition in this order is forced by a positive intersection at the chosen vertex. The result is

$$Z = E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E_6.$$

In the displayed order its intersections are $(0, 0, 0, 0, 0, -1)$: for example $Z \cdot E_3 = -6+2+2+2 = 0$ and $Z \cdot E_6 = -4+3 = -1$. Hence $Z^2 = \sum z_i (Z \cdot E_i) = -2$. The graph has recovered the multiplicity, by Artin's formula, without the use of an equation in three variables.

We can now even predict which curves appear on the *first* blow-up. On the minimal resolution of a rational double point the extension of the maximal ideal is the invertible ideal $\mathcal{O}_X(-Z)$. By the universal property of the blow-up, X maps to $\mathrm{Bl}_y Y$, and the pullback of its relatively ample $\mathcal{O}(1)$ is $\mathcal{O}_X(-Z)$. An exceptional component E_i is therefore contracted by this intermediate map exactly when $-Z \cdot E_i = 0$. For E_6 only the short arm E_6 has positive degree, namely one. The other five curves form the chain $E_1, \ldots, E_5$ and collapse to a point of type A_5. The chart computation with $v^2 + ys^3 + y^2$ and the intersection calculation have independently found the same five curves.

9.5. **What changes in small characteristic.** In characteristic zero and in characteristic $p > 5$ the classical list in Theorem 1.6 gives all the completed equations. The cyclic equation $xy + z^{n+1}$ makes sense even if $p \mid n+1$, although a constant tame cyclic group of that order is no longer available. In small characteristic the exceptional curves still give the same finite ADE diagrams. What fails for some diagrams is *tautness*: the graph can represent more than one formal isomorphism class. The superscript on E_8^r, for example, is the Artin coindex, not the index of the canonical divisor. Every RDP is Gorenstein.

The following is Artin's list for the non-taut types [Art77, Sta21]. For a row with base equation f_0, the completed equations are $f_0 + h = 0$ with the listed corrections in order of increasing coindex (beginning with zero). In the D families use the indicated range for r.

p	Type	f_0	Corrections h
2	D_{2n}	$z^2 + x^2y + xy^n$	$0,\ xy^{n-r}z$
2	D_{2n+1}	$z^2 + x^2y + y^nz$	$0,\ xy^{n-r}z$
2	E_6	$z^2 + x^3 + y^2z$	$0,\ xyz$
2	E_7	$z^2 + x^3 + xy^3$	$0,\ x^2yz,\ y^3z,\ xyz$
2	E_8	$z^2 + x^3 + y^5$	$0,\ xy^3z,\ xy^2z,\ y^3z,\ xyz$
3	E_6	$z^2 + x^3 + y^4$	$0,\ x^2y^2$
3	E_7	$z^2 + x^3 + xy^3$	$0,\ x^2y^2$
3	E_8	$z^2 + x^3 + y^5$	$0,\ x^2y^3,\ x^2y^2$
5	E_8	$z^2 + x^3 + y^5$	$0,\ xy^4$

Here $n \geq 2$ and $1 \leq r \leq n-1$. For types not affected in the indicated characteristic the usual equation applies. For a concrete distinction in characteristic five, $f_0 = z^2 + x^3 + y^5$ has Tjurina algebra $k[[x,y]]/(x^2, y^5)$ of length ten. For $f_1 = f_0 + xy^4$ the partials give $z = 0$, $3x^2 + y^4 = 0$, $xy^3 = 0$. Multiplying the second relation by x gives $x^3 = 0$, hence f_1 gives $y^5 = 0$. A basis is $1, y, \ldots, y^4, x, xy, xy^2$. The length has fallen to eight while the E_8 graph has stayed the same.

Proposition 9.9 (The terms in the small-characteristic tables)**.** *For an isolated formal hypersurface $f = 0$, the space of first-order deformations of its equation modulo changes of coordinates and a unit is*

$$T^1_f = k[[x,y,z]]/(f, f_x, f_y, f_z).$$

For a rational double point the Dynkin type and $\dim_k T^1_f$ *together determine the formal isomorphism class over an algebraically closed field of fixed characteristic.*

Proof. A first-order change is $f + \epsilon g$, with $\epsilon^2 = 0$. Substitution $x_i \mapsto x_i + \epsilon h_i$ changes g by $\sum_i h_i f_{x_i}$, and multiplication by $1 + \epsilon h$ changes it by hf. Conversely every such correction is realized by these substitutions. For the identification with abstract hypersurface deformations see [Sta21, Proposition 2.2]. For the second assertion inspect the following dimensions row by row together with the equation table. Artin's classification supplies completeness; the dimensions and the criterion are recorded in [Sta21, Table 2 and Observation 2.1].

Characteristic	Graph	Dimensions in increasing coindex
2	D_{2n}, D_{2n+1}	$4n$, then $4n-2r$, $1 \le r \le n-1$
2	E_6	$8, 6$
2	E_7	$14, 12, 10, 8$
2	E_8	$16, 14, 12, 10, 8$
3	E_6, E_7	$9, 7$ in each case
3	E_8	$12, 10, 8$
5	E_8	$10, 8$

Even the taut series A_n contains a characteristic effect in the deformation space. In $xy + z^{n+1}$ the first two partials kill y, x in T^1; the last is $(n+1)z^n$. The dimension is n if $p \nmid n+1$ and $n+1$ otherwise. In both cases there is only one analytic type with that graph. □

9.6. **Back on a projective surface.** An RDP del Pezzo surface is a normal projective surface with at worst rational double points and ample $-K_Y$. Its minimal resolution has $-K_X = \pi^*(-K_Y)$ nef and big. Thus it is a *weak del Pezzo surface* and the curves that disappear on the anticanonical model are precisely its (-2)-curves. Indeed a contracted curve has $K_X \cdot C = 0$, and adjunction and negative definiteness identify it as a (-2)-curve. Conversely adjunction gives $K_X \cdot C = 0$ on a (-2)-curve, so ampleness downstairs forces its contraction. On the blow-up of $\mathbb{P}^2$ in r points in almost general position, the degree is $K_X^2 = 9 - r$. Besides such plane models the list of smooth weak del Pezzo surfaces includes $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hirzebruch surface $\mathbb{F}_2$ [Sta21].

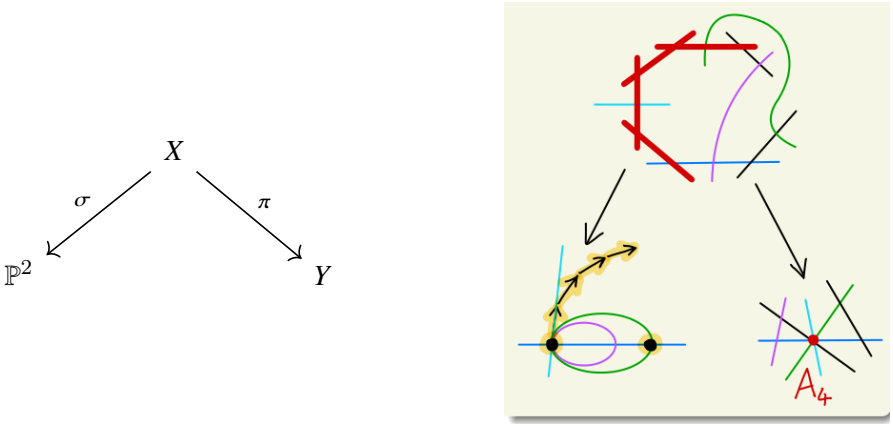


FIGURE 21. Two contractions on a marked weak del Pezzo surface; the drawing on the right is from the concluding lecture. A plane marking σ reverses a sequence of point blow-ups. The anticanonical map to the model Y contracts the configurations of (-2)-curves.

Consider the cubic in $\mathbb{P}^3$ with equation

$$x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_0 = 0.$$

In characteristic zero this is a model of the Clebsch cubic. Each partial derivative is $2x_ix_{i+1} + x_{i-1}^2$, indices read modulo four. If a coordinate of a singular point vanishes then all must vanish. When the coordinates are all nonzero, multiplying the four derivative equations gives $16 = 1$, so only characteristics three and five require further inspection. In characteristic three the derivatives give four equal, nonzero summands in the cubic (multiply each derivative by x_i); their sum is nonzero. In characteristic five the point $[3 : 4 : 2 : 1]$ satisfies all four equations and lies on the surface. It has type A_4; there are six geometric lines on this cubic, compared with twenty-seven on a smooth cubic. We give a check of the local type and of the line count below.

Set $x_3 = 1$ in characteristic five. Write $a = x_0$, $b = x_1$, $c = x_2$; the polynomial is $c^2 + b^2c + a^2b + a$. Completing the square in c gives $(c + b^2/2)^2 + ba^2 + a - b^4/4$. Since b is a unit near the point, complete the square in a as well. The remaining term is

$$-\frac{b^4}{4} - \frac{1}{4b} = -\frac{(b+1)^5}{4b}.$$

The two quadratic variables have unit coefficients, with a square root of b in the completion. Their squares split into a product over k, leaving $UV + (b + 1)^5$ up to a unit and rescaling U. This also shows that the only possible singular point has $b = -1$ and is the one above. The six lines can be read on the standard marked weak cubic: choose a chain of simple roots $E_1 - E_2, \dots, E_4 - E_5$ in its E_6 lattice. A line on the anticanonical model corresponds to a (-1)-class intersecting each contracted simple root nonnegatively. Of the 27 classes E_i, $H - E_i - E_j$ and $2H - \sum_{i \neq j} E_i$ on a marked smooth cubic, these inequalities select E_5, E_6, the two classes with pairs $\{1, 2\}, \{1, 6\}$, and the two conic classes omitting 5 or 6. This recovers six; see the classification of singular cubic surfaces in [Roc96, Dol12].

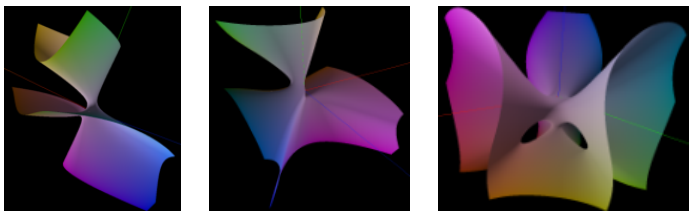
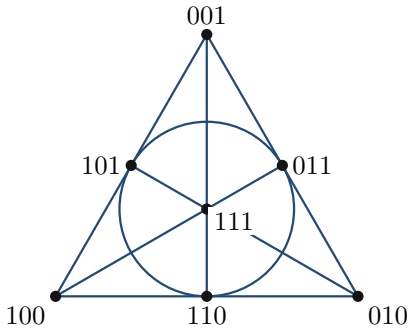


FIGURE 22. Views of the real cubic from the closing lecture. The degeneration in characteristic five is detected by the local calculation above.

Our last surface starts with just seven points. Let k be algebraically closed of characteristic two. In $\mathbb{P}^2_k$ mark the seven points with coordinates in $\mathbb{F}_2$. Every pair lies on a unique $\mathbb{F}_2$-line and every such line contains three marked points: its underlying two-dimensional $\mathbb{F}_2$-vector space has three

nonzero vectors. There are seven lines in all, and three pass through each marked point.



The seven points of $\mathbb{P}^2(\mathbb{F}_2)$, with homogeneous coordinates abbreviated. The circle represents the projective line $x + y + z = 0$.

Blow up the seven points, writing $b : \widetilde{X}_2 \to \mathbb{P}^2_k$, $H = b^*\mathcal{O}_{\mathbb{P}^2}(1)$ for the line class, and E_p for the exceptional curve above p. Denote the strict transform of each marked line ℓ by C_ℓ. The slide shows both maps out of this one surface.

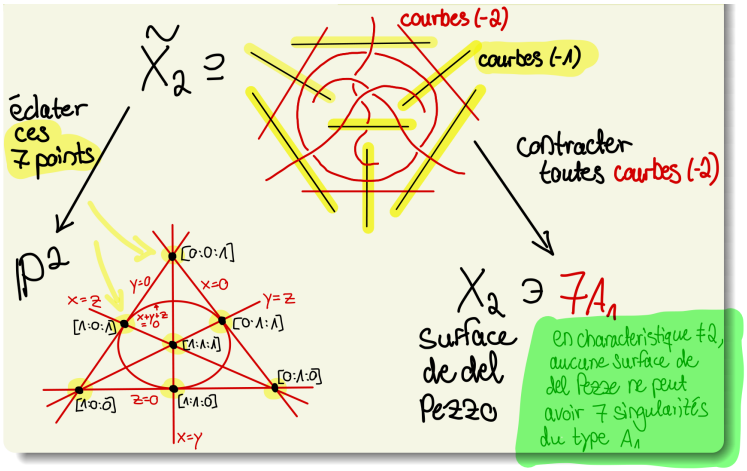


FIGURE 23. The two contractions in the closing lecture (notebook p. 26, slide 11/16). On $\widetilde{X}_2$ the yellow highlights mark the seven (-1)-curves E_p and the red strokes the seven (-2)-curves C_ℓ. The arrow to the plane contracts the E_p. The arrow to X_2 contracts the C_ℓ, each to its own A_1 point.

Follow a single marked line ℓ through the blow-up. The curve C_ℓ is still a smooth rational curve, and the blow-up formula gives

$$C_\ell = H - \sum_{p \in \ell(\mathbb{F}_2)} E_p, \qquad C_\ell^2 = 1 - 3 = -2, \qquad (-K_{\widetilde{X}_2}) \cdot C_\ell = 3 - 3 = 0.$$

For two different lines ℓ, m, the strict transforms have intersection number $1 - \#(\ell \cap m)(\mathbb{F}_2) = 0$. Indeed, blowing up their unique common point separates their tangent directions. On $E_p \simeq \mathbb{P}^1$ the three lines through p now meet E_p transversely at three distinct points. Thus one can still read off the entire Fano incidence pattern on the surface: $E_p \cdot C_\ell = 1$ if $p \in \ell$, and is zero otherwise. The seven C_ℓ alone have intersection matrix $-2I_7$.

Could there be another curve of anticanonical degree zero? Let D be an irreducible plane curve of degree e other than the seven lines, and let m_p be its multiplicity at a marked point. Bézout on each marked line gives $\sum_{p \in \ell(\mathbb{F}_2)} m_p \le e$. Adding the seven inequalities counts every m_p three times. Hence its strict transform satisfies

$$3 \sum_p m_p \le 7e, \qquad (-K_{\widetilde{X}_2}) \cdot \widetilde{D} = 3e - \sum_p m_p \ge 2e/3 > 0.$$

Each E_p has anticanonical degree one. These are all the (-1)-curves on the surface: any further one would be a strict transform $\widetilde{D}$ with anticanonical degree one, so $2e/3 \le 1$ forces $e = 1$. Its line would have to pass through exactly two marked points. The unique line through such a pair already contains a third. This accounts for every irreducible curve on the blow-up, so $-K_{\widetilde{X}_2}$ is nef, and it is big since $K^2_{\widetilde{X}_2} = 9 - 7 = 2$. Its anticanonical model X_2 therefore contracts precisely the seven disjoint C_ℓ. They give seven connected exceptional fibres, each with dual graph a single vertex of weight -2. By Theorem 9.6 these give seven distinct rational double points of type A_1.

Exercise 9.10 (The seven lines as cubics through seven points)**.** On the blow-up of the seven $\mathbb{F}_2$-points, find sections of the anticanonical system and see directly what they do to the line $Z = 0$.

Solution. Write H for the pullback of a line and E_i for the exceptional curves. The blow-up formula gives $-K = 3H - \sum E_i$. The cubics

$$YZ(Y + Z), \qquad ZX(Z + X), \qquad XY(X + Y)$$

vanish at all seven points. To see they span the space of cubics through the points, use the three coordinate points first: they kill the coefficients of X^3, Y^3, Z^3. The three points with precisely two nonzero entries then require the two mixed coefficients involving each such pair of variables to agree. The condition at $[1 : 1 : 1]$ forces the coefficient of XYZ to vanish. This leaves precisely the three displayed independent sections. Their only common zeros are the seven points, because all nonzero coordinate ratios must be one. At a coordinate point their linear parts span the cotangent space, as one sees in the chart $X = 1$ at $Y = Z = 0$; the same is true at every base point by the action of $\mathrm{GL}_3(\mathbb{F}_2)$ on the linear system. A single blow-up therefore separates the directions there. On $Z = 0$ away from the base points the first two cubics vanish identically; the third gives the single

image $[0:0:1]$. Continuity gives the same on the entire strict transform. This calculation exhibits one of the seven contractions directly. The linear system maps to $\mathbb{P}^2$; the degree-two anticanonical *model* is the intermediate normal surface in the Stein factorization. It is the latter surface that has the seven ordinary double points. □

Where does characteristic two enter the picture? The first six lines can be drawn over any field. Choose four of the points with no three collinear. After a projective change of coordinates they are

$$A = [1:0:0], \quad B = [0:1:0], \quad C = [0:0:1], \quad D = [1:1:1].$$

Their six joining lines force the other three points as intersections of opposite sides:

$$U = AB \cap CD = [1:1:0],$$
$$V = AC \cap BD = [1:0:1],$$
$$W = AD \cap BC = [0:1:1],$$

$$\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -2.$$

The seventh line in the picture must pass through U, V, W. The determinant tests exactly this collinearity. It vanishes precisely in characteristic two, when the three points lie on $X + Y + Z = 0$ (the circle in the incidence picture). In any other characteristic the six sides still yield six disjoint (-2)-curves after the blow-ups, but the seventh line is missing.

Proposition 9.11 (Why seven A_1 points are special to characteristic two)**.** *Over an algebraically closed field of characteristic different from two, a degree-two RDP del Pezzo surface has at most six singular points. In characteristic two the blow-up and contraction in Figure 23 attain seven, all of type A_1.*

Proof. To rule out a different construction of seven points, start with an arbitrary degree-two RDP del Pezzo surface Y. Its anticanonical map presents it as a double plane. When 2 is invertible we can complete the square in the weight-two coordinate and write $w^2 = f_4(X, Y, Z)$, with branch quartic $B = V(f_4)$; see [Dol12, §8.7]. The quartic is reduced: a repeated component would make Y singular along a curve, contrary to normality. Locally the partial derivative in w is $2w$, so every singular point lies above a singular point of B, and above each such point there is just one point, with $w = 0$.

Write r for the number of irreducible components of B and g_i for the genera of their normalizations. The normalization formula for plane curves gives

$$\sum_{q \in \mathrm{Sing}(B)} \delta_q = p_a(B) + r - 1 - \sum_i g_i = r + 2 - \sum_i g_i \leq 6.$$

Here $p_a(B) = 3$, $r \leq 4$ and every singular point of a reduced curve contributes at least one. Four lines with no three concurrent attain the bound for the branch curve. This proves the asserted upper bound for every such Y. The construction above proves the characteristic-two assertion. □

The classification also identifies this example uniquely. Over an algebraically closed field, any RDP del Pezzo surface whose entire singularity configuration is $7A_1$ has characteristic two and degree two, and there is just one such surface up to isomorphism [Sta21, Theorem 1.2, Remark 1.3 and Proposition 3.2]. Thus its minimal resolution really does admit the plane model in the lecture drawing.

9.7. **A final look at type A_1.** Finally the same diagrams occur in Lie theory. Over $\mathbb{C}$ a transverse slice to a subregular nilpotent orbit in a simple Lie algebra of simply laced type cuts out the corresponding surface singularity in the nilpotent cone. On the slice the adjoint quotient gives a semiuniversal deformation. After base change from the quotient of a Cartan subalgebra by the Weyl group to the Cartan subalgebra, the family admits a simultaneous resolution [Slo80b, Slo80a].

Already for A_1 one can see both the change of base and the resolution in coordinates. Consider $xy = z^2 - s$ over the s-line. Its central fibre has our familiar double point, whereas every other fibre is smooth. Pull back by $s = t^2$. Set $u = z - t$ and $v = z + t$, so the new total space has equation $xy = uv$ and the parameter is $t = (v-u)/2$. Blow up the ideal (x, u). Its two charts are

$$\begin{aligned} u = xr,\ \ y = rv, &\quad (x, r, v) \longmapsto t = (v - xr)/2, \\ x = uq,\ \ v = qy, &\quad (u, q, y) \longmapsto t = (qy - u)/2. \end{aligned}$$

These are smooth affine three-spaces. On each chart one can eliminate a variable in favour of t, so the family after the blow-up is smooth over the t-line. Away from the origin of $xy = uv$ the ideal is locally principal: if y is a unit then $(x, u) = (u)$, while if v is a unit it equals (x). It is the unit ideal wherever x or u is invertible. The modification is consequently an isomorphism there, in particular on each noncentral fibre.

Above the origin the coordinates r and q are free and satisfy $rq = 1$ on their overlap. The exceptional fibre is thus a single $\mathbb{P}^1$. In the central surface the chart equations become $v = xr$ and $y = xr^2$, or $u = qy$ and $x = q^2y$. One recognizes the same resolution of the quadratic cone that we obtained by blowing up its vertex, with exceptional curve of square -2. The involution $t \mapsto -t$ exchanges u and v and interchanges the two choices of factor. This double covering of the parameter line is the Weyl covering of type A_1. It gives one more reading of the equation from the first lecture: its symmetry is visible even as the singular point disappears in a neighbouring fibre.

End of lecture 12



My poetry–math fun: Rilke poetry associated to each lecture

Originally, I came to mathematics through philosophy; one might even say that if one were to subtract mathematics from philosophy, something would remain that is contained in art and poetry. I am deeply enthusiastic about conversations on these topics, about transdisciplinarity and outreach. I believe that we mathematicians should see ourselves much more as poets. And as a personal passion, I love learning poems by heart and am an enthusiastic admirer of Rainer Maria Rilke.

During this course, in the year of Rilke's 150th birthday, I associated a sonnet with each of the first five meetings. At the Rilke Masterclass at Château Mercier in Sierre I gave a linguistic lecture on the role of geometry as one of the secret "recipes" underlying Rilke's poetry, poetic reading, and his own practice of translation. Perhaps someone reading these notes will find another such connection.

The poems are printed in German after the first edition, published by Insel in Leipzig in 1923 [Ril23]. Rilke wrote the fifty-five sonnets of the two parts in February 1922 at Château de Muzot in Switzerland, during the same creative period as the *Duino Elegies*. The pairings for lectures 6–12 were chosen for the written edition.

Date in 2025	Lecture	Sonnet
24 September	Platonic solids and their invariants	II, 12
1 October	Affine algebraic geometry	II, 3
8 October	Local rings and dimension	I, 3
15 October	Tangent spaces and completion	I, 9
22 October	Formal changes of coordinates	II, 4
29 October	Plane curve double points	I, 6
5 November	Directions and blow-ups	II, 18
12 November	The charts of a blow-up	I, 4
19 November	Normalization and resolution	I, 11
26 November	Cones and valuations	I, 1
3 December	Divisors and the canonical class	II, 1
10 December	The ADE characterization	I, 12

Lecture 1 *Sonnets to Orpheus, Part II, Sonnet 12*

Wolle die Wandlung. O sei für die Flamme begeistert,
drin sich ein Ding dir entzieht, das mit Verwandlungen prunkt;
jener entwerfende Geist, welcher das Irdische meistert,
liebt in dem Schwung der Figur nichts wie den wendenden Punkt.

Was sich ins Bleiben verschließt, schon ists *das Erstarrte;*
wähnt es sich sicher im Schutz des unscheinbaren Grau's?
Warte, ein Härtestes warnt aus der Ferne das Harte.
Wehe –: abwesender Hammer holt aus!

Wer sich als Quelle ergießt, den erkennt die Erkennung;
und sie führt ihn entzückt durch das heiter Geschaffne,
das mit Anfang oft schließt und mit Ende beginnt.

Jeder glückliche Raum ist Kind oder Enkel von Trennung,
den sie staunend durchgehn. Und die verwandelte Daphne
will, seit sie lorbeern fühlt, daß du dich wandelst in Wind

Lecture 2 *Sonnets to Orpheus, Part II, Sonnet 3*

Spiegel: noch nie hat man wissend beschrieben,
was ihr in euerem Wesen seid.
Ihr, wie mit lauter Löchern von Sieben
erfüllten Zwischenräume der Zeit.

Ihr, noch des leeren Saales Verschwender –,
wenn es dämmert, wie Wälder weit . . .
Und der Lüster geht wie ein Sechzehn-Ender
durch eure Unbetretbarkeit.

Manchmal seid ihr voll Malerei.
Einige scheinen in euch gegangen –,
andere schicktet ihr scheu vorbei.

Aber die Schönste wird bleiben, bis
drüben in ihre enthaltenen Wangen
eindrang der klare gelöste Narziß.

Lecture 3 *Sonnets to Orpheus, Part I, Sonnet 3*

Ein Gott vermags. Wie aber, sag mir, soll
ein Mann ihm folgen durch die schmale Leier?
Sein Sinn ist Zwiespalt. An der Kreuzung zweier
Herzwege steht kein Tempel für Apoll.

Gesang, wie du ihn lehrst, ist nicht Begehr,
nicht Werbung um ein endlich noch Erreichtes;
Gesang ist Dasein. Für den Gott ein Leichtes.
Wann aber sind wir? Und wann wendet er

an unser Sein die Erde und die Sterne?
Dies ists nicht, Jüngling, daß du liebst, wenn auch
die Stimme dann den Mund dir aufstößt, – lerne

vergessen, daß du aufsangst. Das verrinnt.
In Wahrheit singen, ist ein andrer Hauch.
Ein Hauch um nichts. Ein Wehn im Gott. Ein Wind.

Lecture 4 *Sonnets to Orpheus, Part I, Sonnet 9*

Nur wer die Leier schon hob
auch unter Schatten,
darf das unendliche Lob
ahnend erstatten.

Nur wer mit Toten vom Mohn
aß, von dem ihren,
wird nicht den leisesten Ton
wieder verlieren.

Mag auch die Spieglung im Teich
oft uns verschwimmen:
Wisse das Bild.

Erst in dem Doppelbereich
werden die Stimmen
ewig und mild.

Lecture 5 *Sonnets to Orpheus, Part II, Sonnet 4*

O dieses ist das Tier, das es nicht gibt.
Sie wußtens nicht und habens jeden Falls
– sein Wandeln, seine Haltung, seinen Hals,
bis in des stillen Blickes Licht – geliebt.

Zwar war *es nicht. Doch weil sie's liebten, ward*
ein reines Tier. Sie ließen immer Raum.
Und in dem Raume, klar und ausgespart,
erhob es leicht sein Haupt und brauchte kaum

zu sein. Sie nährten es mit keinem Korn,
nur immer mit der Möglichkeit, es sei.
Und die gab solche Stärke an das Tier,

daß es aus sich ein Stirnhorn trieb. Ein Horn.
Zu einer Jungfrau kam es weiß herbei –
und war im Silber-Spiegel und in ihr.

Lecture 6 *Sonnets to Orpheus, Part I, Sonnet 6*

Ist er ein Hiesiger? Nein, aus beiden
Reichen erwuchs seine weite Natur.
Kundiger böge die Zweige der Weiden,
wer die Wurzeln der Weiden erfuhr.

Geht ihr zu Bette, so laßt auf dem Tische
Brot nicht und Milch nicht; die Toten ziehts –.
Aber er, der Beschwörende, mische
unter der Milde des Augenlids

ihre Erscheinung in alles Geschaute;
und der Zauber von Erdrauch und Raute
sei ihm so wahr wie der klarste Bezug.

Nichts kann das gültige Bild ihm verschlimmern;
sei es aus Gräbern, sei es aus Zimmern,
rühme er Fingerring, Spange und Krug.

Lecture 7 *Sonnets to Orpheus, Part II, Sonnet 18*

Tänzerin: o du Verlegung
alles Vergehens in Gang: wie brachtest du's dar.
Und der Wirbel am Schluß, dieser Baum aus Bewegung,
nahm er nicht ganz in Besitz das erschwungene Jahr?

Blühte nicht, daß ihn dein Schwingen von vorhin umschwärme,
plötzlich sein Wipfel von Stille? Und über ihr,
war sie nicht Sonne, war sie nicht Sommer, die Wärme,
diese unzählige Wärme aus dir?

Aber er trug auch, er trug, dein Baum der Ekstase.
Sind sie nicht seine ruhigen Früchte: der Krug,
reifend gestreift, und die gereiftere Vase?

Und in den Bildern: ist nicht die Zeichnung geblieben,
die deiner Braue dunkler Zug
rasch an die Wandung der eigenen Wendung geschrieben?

Lecture 8 *Sonnets to Orpheus, Part I, Sonnet 4*

O ihr Zärtlichen, tretet zuweilen
in den Atem, der euch nicht meint,
laßt ihn an eueren Wangen sich teilen,
hinter euch zittert er, wieder vereint.

O ihr Seligen, o ihr Heilen,
die ihr der Anfang der Herzen scheint.
Bogen der Pfeile und Ziele von Pfeilen,
ewiger glänzt euer Lächeln verweint.

Fürchtet euch nicht zu leiden, die Schwere,
gebt sie zurück an der Erde Gewicht;
schwer sind die Berge, schwer sind die Meere.

Selbst die als Kinder ihr pflanztet, die Bäume,
wurden zu schwer längst; ihr trüget sie nicht.
Aber die Lüfte . . . aber die Räume . . .

Lecture 9 *Sonnets to Orpheus, Part I, Sonnet 11*

Sieh den Himmel. Heißt kein Sternbild „Reiter“?
Denn dies ist uns seltsam eingeprägt:
dieser Stolz aus Erde. Und ein zweiter,
der ihn treibt und hält und den er trägt.

Ist nicht so, gejagt und dann gebändigt,
diese sehnige Natur des Seins?
Weg und Wendung. Doch ein Druck verständigt.
Neue Weite. Und die zwei sind eins.

Aber sind *sie's? Oder meinen beide*
nicht den Weg, den sie zusammen tun?
Namenlos schon trennt sie Tisch und Weide.

Auch die sternische Verbindung trügt.
Doch uns freue eine Weile nun,
der Figur zu glauben. Das genügt

Lecture 10 *Sonnets to Orpheus, Part I, Sonnet 1*

Da stieg ein Baum. O reine Übersteigung!
O Orpheus singt! O hoher Baum im Ohr!
Und alles schwieg. Doch selbst in der Verschweigung
ging neuer Anfang, Wink und Wandlung vor.

Tiere aus Stille drangen aus dem klaren
gelösten Wald von Lager und Genist;
und da ergab sich, daß sie nicht aus List
und nicht aus Angst in sich so leise waren,

sondern aus Hören. Brüllen, Schrei, Geröhr
schien klein in ihren Herzen. Und wo eben
kaum eine Hütte war, dies zu empfangen,

ein Unterschlupf aus dunkelstem Verlangen
mit einem Zugang, dessen Pfosten beben, –
da schufst du ihnen Tempel im Gehör.

Lecture 11 *Sonnets to Orpheus, Part II, Sonnet 1*

Atmen, du unsichtbares Gedicht!
Immerfort um das eigne
Sein rein eingetauschter Weltraum. Gegengewicht,
in dem ich mich rhythmisch ereigne.

Einzige Welle, deren
allmähliches Meer ich bin;
sparsamstes du von allen möglichen Meeren, –
Raumgewinn.

Wie viele von diesen Stellen der Räume waren schon
innen in mir. Manche Winde
sind wie mein Sohn.

Erkennst du mich, Luft, du, voll noch einst meiniger Orte?
Du, einmal glatte Rinde,
Rundung und Blatt meiner Worte.

Lecture 12 *Sonnets to Orpheus, Part I, Sonnet 12*

Heil dem Geist, der uns verbinden mag;
denn wir leben wahrhaft in Figuren.
Und mit kleinen Schritten gehn die Uhren
neben unserm eigentlichen Tag.

Ohne unsern wahren Platz zu kennen,
handeln wir aus wirklichem Bezug.
Die Antennen fühlen die Antennen,
und die leere Ferne trug . . .

Reine Spannung. O Musik der Kräfte!
Ist nicht durch die läßlichen Geschäfte
jede Störung von dir abgelenkt?

Selbst wenn sich der Bauer sorgt und handelt,
wo die Saat in Sommer sich verwandelt,
reicht er niemals hin. Die Erde schenkt.

References

[Abh56] S. Abhyankar. Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$. *Ann. of Math. (2)*, 63:491–526, 1956.

[Alb24] G. Albanese. Trasformazione birazionale di una superficie algebrica qualunque in un'altra priva di punti multipli. *Rend. Circ. Mat. Palermo*, 48:321–332, 1924.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Art66] M. Artin. On isolated rational singularities of surfaces. *Amer. J. Math.*, 88:129–136, 1966.

[Art77] M. Artin. Coverings of the rational double points in characteristic p. In *Complex analysis and algebraic geometry*, pages 11–22. Iwanami Shoten, Tokyo, 1977.

[Bea10] Arnaud Beauville. Finite subgroups of $\mathrm{PGL}_2(K)$. In *Vector bundles and complex geometry*, volume 522 of *Contemporary Mathematics*, pages 23–29. American Mathematical Society, 2010.

[Bou89] Nicolas Bourbaki. *Commutative Algebra. Chapters 1–7*. Springer-Verlag, Berlin, 1989.

[CP09] Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic. II. *J. Algebra*, 321:1836–1976, 2009.

[Dol12] I. V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view.

[Dur79] A. H. Durfee. Fifteen characterizations of rational double points and simple critical points. *Enseign. Math. (2)*, 25(1-2):131–163, 1979.

[DV34] P. Du Val. On isolated singularities of surfaces which do not affect the conditions of adjunction, Parts I, II, III. *Proc. Cambridge Phil. Soc.*, 30:453–459; 460–465; 483–491, 1934.

[Enr49] F. Enriques. *Le superficie algebriche*. Nicola Zanichelli, Bologna, 1949.

[Euc56] Euclid. *The thirteen books of Euclid's Elements translated from the text of Heiberg. Vol. I: Introduction and Books I, II. Vol. II: Books III–IX. Vol. III: Books X–XIII and Appendix*. Dover Publications, Inc., New York, 1956. Translated with introduction and commentary by Thomas L. Heath, 2nd ed.

[FH10] Eleonore Faber and Herwig Hauser. Today's menu: geometry and resolution of singular algebraic surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 47(3):373–417, 2010.

[GK90] G.-M. Greuel and H. Kröning. Simple singularities in positive characteristic. *Mathematische Zeitschrift*, 203(3):339–354, 1990.

[Gro71] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin–New York, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–61.

[Har77] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)*, 79:109–203, 205–326, 1964.

[Kep96] J. Kepler. *Prodromus dissertationum cosmographicarum, continens mysterium cosmographicum, de admirabili proportione orbium coelestium [...]*. Tübingen, 1596.

[Kle93] F. Klein. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Birkhäuser Verlag, Basel; B. G. Teubner, Stuttgart, 1993. Reprint of the 1884 original, edited, with an introduction and commentary by Peter Slodowy.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1998.

[Lam86] K. Lamotke. *Regular Solids and Isolated Singularities*. Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 1986.

[Lev97] B. Levi. Risoluzione delle singolarità puntuali delle superficie algebriche. *Atti Acc. Sc. Torino*, 33:66–86, 1897.

[Lip69] J. Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. *Inst. Hautes Études Sci. Publ. Math.*, 36:195–279, 1969.

[Lip75] Joseph Lipman. Introduction to resolution of singularities. In *Algebraic geometry (Arcata, Calif., 1974)*, volume 29 of *Proc. Sympos. Pure Math.*, pages 187–230. American Mathematical Society, Providence, RI, 1975.

[Lip78] Joseph Lipman. Desingularization of two-dimensional schemes. *Ann. of Math. (2)*, 107:151–207, 1978.

[Mum61] David Mumford. The topology of normal singularities of an algebraic surface and a criterion for simplicity. *Publ. Math. Inst. Hautes Études Sci.*, 9:5–22, 1961.

[Pla29] Plato. *Timaeus, Critias, Cleitophon, Menexenus, Epistles*, volume 234 of *Loeb Classical Library*. William Heinemann, 1929. Translated by R. G. Bury.

[Ril06] Rainer Maria Rilke. *Das Buch der Bilder*. Axel Juncker, Berlin, second, enlarged edition, 1906.

[Ril23] Rainer Maria Rilke. *Die Sonette an Orpheus. Geschrieben als ein Grab-Mal für Wera Ouckama Knoop*. Insel-Verlag, Leipzig, 1923. First edition. Text and facsimile.

[Roc96] M. Roczen. Cubic surfaces with double points in positive characteristic. In *Algebraic geometry and singularities (La Rábida, 1991)*, volume 134 of *Progr. Math.*, pages 375–382. Birkhäuser, Basel, 1996.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Seg97] C. Segre. Sulla scomposizione dei punti singolari delle superficie algebriche. *Annali di Mat. pura appl. (2)*, 25:2–54, 1897.

[Sev14] F. Severi. Trasformazione birazionale di una superficie algebrica qualunque, in una priva di punte multiple. *Rend. Acc. Lincei Rom. (5)*, 23(2):527–539, 1914.

[Slo80a] P. Slodowy. *Four lectures on simple groups and singularities*, volume 11 of *Communications of the Mathematical Institute, Rijksuniversiteit Utrecht*. Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1980.

[Slo80b] P. Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Slo83] P. Slodowy. Platonic solids, Kleinian singularities, and Lie groups. In *Algebraic geometry (Ann Arbor, Mich., 1981)*, volume 1008 of *Lecture Notes in Math.*, pages 102–138. Springer, Berlin, 1983.

[Sta21] C. Stadlmayr. Which rational double points occur on del Pezzo surfaces? *Épijournal Géom. Algébrique*, 5:1–27, 2021. Article 17.

[Sta26] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2026.

[Tei07] Bernard Teissier. Complex curve singularities: A biased introduction. In *Singularities in Geometry and Topology*, pages 825–887. World Scientific, 2007.

[Wal35] R. J. Walker. Reduction of the singularities of an algebraic surface. *Ann. of Math. (2)*, 36(2):336–365, 1935.

[Wal04] C. T. C. Wall. *Singular Points of Plane Curves*, volume 63 of *London Mathematical Society Student Texts*. Cambridge University Press, 2004.

[Wat72] W. C. Waterhouse. The discovery of the regular solids. *Arch. History Exact Sci.*, 9(3):212–221, 1972.

[Yau78] Shing-Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

[Zar39] O. Zariski. The reduction of the singularities of an algebraic surface. *Ann. of Math. (2)*, 40:639–689, 1939.

Aus unendlichen Sehnsüchten steigen
endliche Taten wie schwache Fontänen,
die sich zeitig und zitternd neigen.
Aber, die sich uns sonst verschweigen,
unsere fröhlichen Kräfte – zeigen
sich in diesen tanzenden Tränen.

Rainer Maria Rilke, «Initiale», *Das Buch der Bilder* [Ril06].

Current address: Institut de Mathématiques – Université de Neuchâtel, Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland

Email address: claudia.stadlmayr@unine.ch